# Adaptive Control Design and Analysis

*(Supplemental Notes)*

**Gang Tao**

Department of Electrical and Computer Engineering
University of Virginia
Charlottesville, VA 22903

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# Matrix Theory

## Symmetric Matrices

For a matrix $M = M^T$, we have $M = \sum_{i=1}^{n} \lambda_i e_i e_i^T$ where $\lambda_i$ and $e_i$ are the eigenvalues and eigenvectors of $M$ such that $e_i^T e_i = 1$ and $e_i^T e_j = 0$ with $i \neq j$. With $P = [e_1, e_2, \ldots, e_n]$ and $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, it follows that $M = P\Lambda P^T$, where $PP^T = P^T P = I$, and, in addition, for $M$ nonsingular, that $M^{-1} = P\Lambda^{-1}P^T$.

For $M = M^T \geq 0$, we define $M^{1/2} = \sum_{i=1}^{n} \sqrt{\lambda_i} e_i e_i^T = P\Lambda^{1/2}P^T$ and express $M = M^{1/2}M^{1/2}$, where $(M^{1/2})^T = M^{1/2}$. It also follows that for $M$ nonsingular, $(M^{1/2})^{-1} = \sum_{i=1}^{n} \frac{1}{\sqrt{\lambda_i}} e_i e_i^T = P\Lambda^{-1/2}P^T$. On the other hand, for $Q = P\Lambda^{1/2}$, we have $M = QQ^T$ (as compared with $M = M^{1/2}M^{1/2}$ for $(M^{1/2})^T = M^{1/2} = P\Lambda^{1/2}P^T$).

## Singular Values and Eigenvalues of A Matrix

Let the singular values of a square matrix $M \in R^{n\times n}$ be $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n_\theta} \geq 0$ and the absolute eigenvalues of $M$ be $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n_\theta}| \geq 0$. Then, from page 347 of Horn and Johnson (2013) (Horn, R. A. and C. R. Johnson, *Matrix Analysis*, 2nd Ed., Cambridge University Press, 2013), we have that $|\lambda_1| \leq \sigma_1$, and that if $M$ is nonsingular, then $|\lambda_n| \geq \sigma_n > 0$.

# Radial Unboundedness Condition on $V(x)$ for Asymptotic Stability

One can draw the surface plots of $V(x) = c$, for different values of $c$, with

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2. \tag{1}$$

For this $V(x)$, such surface plots are closed curves for $c < 1$ but are open curves for $c > 1$, as $V(x)$ is not radially unbounded.

One can draw the phase-plane plot of $\dot{x} = f(x)$, by obtaining its numerical solutions for some typical initial conditions, for

$$\dot{x}_1 = -\frac{6x_1}{(1+x_1^2)^2} + 2x_2,\ \dot{x}_2 = -\frac{2(x_1+x_2)}{(1+x_1^2)^2}. \tag{2}$$

For some initial conditions, the solution trajectories do not converge to the origin. (Are there any trajectories going to $\infty$?)

One can draw the vector field of $\dot{x}_1 = -\frac{6x_1}{(1+x_1^2)^2} + 2x_2$, $\dot{x}_2 = -\frac{2(x_1+x_2)}{(1+x_1^2)^2}$.

The solution $x = [x_1, x_2]^T$ of this system on the hyperbola $x_2 = \frac{2}{x_1 - \sqrt{2}}$ satisfies

$$g_1(x) = \frac{\dot{x}_2}{\dot{x}_1} = -\frac{1}{1+2\sqrt{2}x_1+2x_1^2}, \tag{3}$$

while the slope of this hyperbola is

$$g_2(x) = \frac{dx_2}{dx_1} = -\frac{1}{1-2\sqrt{2}x_1+\frac{x_1^2}{2}}. \tag{4}$$

It follows that $0 > g_1(x) > g_2(x)$ for $x_1 > \sqrt{2}$ so that the trajectories to the right of the hyperbola branch in the first quadrant cannot cross the branch.

This example shows that for asymptotic stability, the radial unboundedness of $V(x)$ is a crucial condition [179], [351], [426].

# Passivity of A Mass-Damper-Spring System

Consider a mass-damper-spring mechanical system with equation of motion

$$M\ddot{x} + D\dot{x} + Kx = F, \tag{5}$$

where $M$ is the mass, $D$ is the damping constant, $K$ is the spring constant, $x$ is the mass position, and $F$ is the force acting on the mass.

The system energy is

$$V(x,\dot{x}) = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Kx^2 \tag{6}$$

whose time derivative is

$$\frac{d}{dt}V(x,\dot{x}) = F\dot{x} - D\dot{x}^2. \tag{7}$$

Over any time interval $[0,T]$, it follows that

$$V(x(T),\dot{x}(T)) = V(x(0),\dot{x}(0)) + \int_0^T F(t)\dot{x}(t)\,dt - \int_0^T D\dot{x}^2(t)\,dt. \tag{8}$$

Since $D \geq 0$, we have

$$-\int_0^T F(t)\dot{x}(t)\,dt \leq V(x(0),\dot{x}(0)), \tag{9}$$

which means that the energy extracted from the system is less than or equal to the initial system energy. From (7), the term $F\dot{x}$ clearly represents the system absorbed power from the input force $F$. For passivity analysis, the product of system input and output is defined as such power. In this sense we consider the velocity $v = \dot{x}$ as system output. Then, the system admittance (the reciprocal of impedance) is

$$G(s) = \frac{V(s)}{F(s)} = \frac{s}{Ms^2 + Ds + K}, \tag{10}$$

which is positively real (the mechanical-electric analogy pairs are "force vs. voltage" and "velocity vs. current").

If we consider $x$ as system output, the term $Fx$ does not represents system power (i.e., $\int_0^T F(t)x(t)\,dt$ does not represents system energy) so that the passivity analysis is not applicable. In other words, the system transfer function for passivity analysis in terms of positive realness is defined in terms of system impedance (or admittance) relating current (velocity) to voltage (force) (or voltage (force) to current (velocity)).

Without the passivity property, the transfer function from the input force to the output position, $\frac{1}{Ms^2+Ds+K}$, cannot be positive real.

# Positive Real Functions

A popular definition of positive real (PR) functions is that a function $h(s)$ of the complex variable $s = \sigma + j\omega$ is positive real if (i) $h(s)$ is real for real $s$, and (ii) $Re[h(s)] \geq 0$ for all $s$ such that $\mathrm{Re}[s] > 0$.

One may "induce" a definition of strictly positive real (SPR) functions as: a function $h(s)$ of the complex variable $s = \sigma + j\omega$ is strictly positive real if (i) $h(s)$ is real for real $s$, and (ii) $Re[h(s)] > 0$ for all $s$ such that $\mathrm{Re}[s] > 0$. This definition was once used in the early literature for SPR functions. It turns out that, like some other definitions of SPR functions, this definition does not capture the physical meaning of strictly positive realness, as indicated by the following example.

It is well-understood that a proper definition of strictly positive functions, which captures the physical meaning of strictly positive realness, is that $h(s)$ is strictly positive real if $h(s-\varepsilon)$ is positive real for some $\varepsilon > 0$. Based on this definition,

$$h(s) = \frac{s+1}{s^2+s+1} \tag{11}$$

is only positive real but not strictly positive real. From the expressions

$$\begin{aligned} h(s) &= \frac{\sigma + j\omega + 1}{(\sigma + j\omega)^2 + \sigma + j\omega + 1} \\ &= \frac{\sigma + 1 + j\omega}{\sigma^2 - \omega^2 + \sigma + 1 + j(\omega + 2\sigma\omega)} \\ &= \frac{(\sigma + 1 + j\omega)(\sigma^2 - \omega^2 + \sigma + 1 - j(\omega + 2\sigma\omega))}{(\sigma^2 - \omega^2 + \sigma + 1)^2 + (\omega + 2\sigma\omega)^2} \end{aligned} \tag{12}$$

$$\begin{aligned} \mathrm{Re}[h(s)] &= \frac{(\sigma + 1)(\sigma^2 - \omega^2 + \sigma + 1) + \omega(\omega + 2\sigma\omega)}{(\sigma^2 - \omega^2 + \sigma + 1)^2 + (\omega + 2\sigma\omega)^2} \\ &= \frac{\sigma^3 + 2\sigma^2 + \sigma\omega^2 + 2\sigma + 1}{(\sigma^2 - \omega^2 + \sigma + 1)^2 + (\omega + 2\sigma\omega)^2}, \end{aligned} \tag{13}$$

we see that $h(s)$ satisfies the above "induced" definition of SPR functions: (i) $h(s)$ is real for real $s$, and (ii) $Re[h(s)] > 0$ for all $s$ such that $\mathrm{Re}[s] = \sigma > 0$. Hence, the conclusion is that this "induced" definition for SPR functions is not proper.

Note that this $h(s)$ is not SPR, because

$$\mathrm{Re}[h(j\omega)] = \frac{1}{(1-\omega^2)^2 + \omega^2} \tag{14}$$

does not satisfy the necessary condition for SPRness: $\lim_{\omega^2 \to \infty} \omega^2 \,\mathrm{Re}[h(j\omega)] > 0$, or because for any chosen $\varepsilon > 0$,

$$\mathrm{Re}[h(j\omega - \varepsilon)] = \frac{-\varepsilon^3 + 2\varepsilon^2 - \varepsilon\omega^2 - 2\varepsilon + 1}{(\varepsilon^2 - \omega^2 - \varepsilon + 1)^2 + (\omega - 2\varepsilon\omega)^2} < 0 \tag{15}$$

whenever $\omega^2 > (-\varepsilon^3 + 2\varepsilon^2 - 2\varepsilon + 1)/\varepsilon$, that is, $h(s-\varepsilon)$ cannot be positive real for any $\varepsilon > 0$. ($P(\varepsilon) = -\varepsilon^3 + 2\varepsilon^2 - 2\varepsilon + 1 = (\varepsilon - 1)(-\varepsilon^2 + \varepsilon - 1) > 0$ for $\varepsilon \in (0,1)$ and $P(\varepsilon) < 0$ for $\varepsilon > 1$. For $h(s-\varepsilon)$ to be stable, $\varepsilon \in [0, 0.5)$ is needed as $(s-\varepsilon)^2 + s - \varepsilon + 1 = s^2 + (1-2\varepsilon)s + 1 - \varepsilon + \varepsilon^2$ and $1 - \varepsilon + \varepsilon^2 > 0$ for any $\varepsilon$.)

# Parameter Projection Properties

The property (3.183) follows from the observation: if $\theta_j(t) = \theta_j^a$ and $g_j(t) < 0$, then $f_j(t) = -g_j(t) > 0$ and $\theta_j(t) - \theta_j^* = \theta_j^a - \theta_j^* \leq 0$, so that $(\theta_j(t) - \theta_j^*) f_j(t) \leq 0$; and if $\theta_j(t) = \theta_j^b$ and $g_j(t) > 0$, then $f_j(t) = -g_j(t) < 0$ and $\theta_j(t) - \theta_j^* = \theta_j^b - \theta_j^* \geq 0$, so that $(\theta_j(t) - \theta_j^*) f_j(t) \leq 0$.

Similarly, the property (3.214) follows from the observation: if $\theta_j(t) + g_j(t) > \theta_j^b$, then $f_j(t) = \theta_j^b - \theta_j(t) - g_j(t) < 0$ and $\theta_j(t) - \theta_j^* + g_j(t) + f_j(t) = \theta_j^b - \theta_j^* \geq 0$, so that $f_j(t)(\theta_j(t) - \theta_j^* + g_j(t) + f_j(t)) \leq 0$; and if if $\theta_j(t) + g_j(t) < \theta_j^a$, then $f_j(t) = \theta_j^b - \theta_j(t) - g_j(t) > 0$ and $\theta_j(t) - \theta_j^* + g_j(t) + f_j(t) = \theta_j^a - \theta_j^* \leq 0$, so that $f_j(t)(\theta_j(t) - \theta_j^* + g_j(t) + f_j(t)) \leq 0$.

In the case when a parameter component $\theta_j^*$ is known, we have $\theta_j^a = \theta_j^b$. This means that we can simply set $\theta_j = \theta_j^*$ if $\theta_j^*$ is known, but still with the use of a diagonal $\Gamma > 0$ or $\Gamma = \mathrm{diag}\{\Gamma_1, \gamma_j, \Gamma_2\} = \Gamma^T > 0$ with $\Gamma_1 \in R^{(j-1)\times(j-1)}$, $\gamma_j \in R$ and $\Gamma_2 \in R^{(n_\theta - j)\times(n_\theta - j)}$, if $\theta_j$ is the only component to be projected.

# Explicit Swapping Lemma

The swapping lemma (5.331) was an important lemma in the development of a stable model reference adaptive control system. It states that for a stable and proper rational function $h(s)$ with a minimal realization $h(s) = c(sI-A)^{-1}b+d$ and two vector signals $\theta(t)$ and $\omega(t)$, it follows that

$$\theta^T(t)h(s)[\omega](t) - h(s)[\theta^T\omega](t) = h_c(s)[h_b(s)[\omega^T]\dot{\theta}](t), \tag{16}$$

where $h_c(s) = c(sI-A)^{-1}$ and $h_b(s) = (sI-A)^{-1}b$. Here we derive an alternative form of this lemma, explicitly in terms of the parameters of the function $h(s)$.

Denoting $P_m(s) = s^{n^*} + a_{n^*-1}s^{n^*-1} + \cdots + a_1 s + a_0$, for vector signals $\theta(t)$ and $\omega(t)$, from (5.138) and with $a_{n^*} = 1$, we have

$$\begin{aligned} &\theta^T(t)\frac{1}{P_m(s)}[\omega](t) - \frac{1}{P_m(s)}[\theta^T\omega](t) \\ &= \sum_{i=1}^{n^*}\left(\frac{\sum_{j=0}^{n^*-i} a_{n^*-j}s^{n^*-i-j}}{P_m(s)}\right)\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right](t). \end{aligned} \tag{17}$$

Introducing $F(s) = f_{n^*-1}s^{n^*-1} + \cdots + f_1 s + f_0$, we express

$$\begin{aligned} &F(s)\left[\theta^T(t)\frac{1}{P_m(s)}[\omega]\right](t) \\ &= \left(f_{n^*-1}s^{n^*-2} + \cdots + f_1\right)\left[\dot{\theta}^T\frac{1}{P_m(s)}[\omega] + \theta^T\frac{s}{P_m(s)}[\omega]\right](t) + \theta^T(t)\frac{f_0}{P_m(s)}[\omega](t) \\ &= \left(f_{n^*-1}s^{n^*-2} + \cdots + f_1\right)\left[\dot{\theta}^T\frac{1}{P_m(s)}[\omega]\right](t) \\ &\quad + \left(f_{n^*-1}s^{n^*-3} + \cdots + f_2\right)\left[\dot{\theta}^T\frac{s}{P_m(s)}[\omega] + \theta^T\frac{s^2}{P_m(s)}[\omega]\right](t) \\ &\quad + \theta^T(t)\frac{f_1 s + f_0}{P_m(s)}[\omega](t) = \cdots \\ &= \sum_{i=1}^{n^*-1}\left(\sum_{j=1}^{n^*-i} f_{n^*-j}s^{n^*-i-j}\right)\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right](t) + \theta^T(t)\frac{F(s)}{P_m(s)}[\omega](t) \end{aligned} \tag{18}$$

and use it to derive

$$\begin{aligned} &\frac{F(s)}{P_m(s)}[\theta^T\omega](t) - \theta^T(t)\frac{F(s)}{P_m(s)}[\omega](t) \\ &= \sum_{i=1}^{n^*-1}\left(\sum_{j=1}^{n^*-i} f_{n^*-j}s^{n^*-i-j}\right)\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right](t) \end{aligned}$$

$$
\begin{aligned}
&- F(s)\left[\sum_{i=1}^{n^*}\frac{\sum_{j=0}^{n^*-i} a_{n^*-j}s^{n^*-i-j}}{P_m(s)}\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right]\right](t)\\
&= \sum_{i=1}^{n^*-1}\left(\sum_{j=1}^{n^*-i} f_{n^*-j}s^{n^*-i-j} - \frac{F(s)\sum_{j=0}^{n^*-i} a_{n^*-j}s^{n^*-i-j}}{P_m(s)}\right)\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right](t)\\
&- \frac{F(s)}{P_m(s)}\left[\dot{\theta}^T\frac{s^{n^*-1}}{P_m(s)}[\omega]\right](t)\\
&= \sum_{i=1}^{n^*-1}\frac{\alpha_i(s)}{P_m(s)}\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right](t) - \frac{F(s)}{P_m(s)}\left[\dot{\theta}^T\frac{s^{n^*-1}}{P_m(s)}[\omega]\right](t). \qquad (19)
\end{aligned}
$$

Hence, we obtain the *explicit swapping lemma*:

$$
\frac{F(s)}{P_m(s)}[\theta^T\omega](t) - \theta^T(t)\frac{F(s)}{P_m(s)}[\omega](t) = \sum_{i=1}^{n^*}\frac{\alpha_i(s)}{P_m(s)}\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right](t), \qquad (20)
$$

where

$$
\alpha_i(s) = P_m(s)\sum_{j=1}^{n^*-i} f_{n^*-j}s^{n^*-i-j} - F(s)\sum_{j=0}^{n^*-i} a_{n^*-j}s^{n^*-i-j},\ i = 1,2,\ldots,n^*-1, \qquad (21)
$$

are polynomials of degrees $n^*-1$ or less, and $\alpha_{n^*}(s) \stackrel{\triangle}{=} -F(s)$.

For a proper transfer function $h(s) = f_{n^*} + \frac{F(s)}{P_m(s)}$, it follows that

$$
h(s)[\theta^T\omega](t) - \theta^T(t)h(s)[\omega](t) = \sum_{i=1}^{n^*}\frac{\alpha_i(s)}{P_m(s)}\left[\dot{\theta}^T\frac{s^{i-1}}{P_m(s)}[\omega]\right](t). \qquad (22)
$$

# Discrete-Time Swapping Lemma

Let a stable and proper rational function $h(z)$ have a minimal realization $h(z) = c(zI-A)^{-1}b+d$ and $\theta(t)$ and $\omega(t)$ be two vector signals, and denote $h_c(z) = c(zI-A)^{-1}$ and $h_b(z) = (zI-A)^{-1}b$. Then,

$$\theta^T(t)h(z)[\omega](t) - h(z)[\theta^T\omega](t) = h_c(z)[(h_b(z)z)[\omega^T](z-1)[\theta]](t). \tag{23}$$

**Proof**: Using the discrete-time convolution: $y(t) = C\sum_{i=0}^{t-1}A^{t-i-1}Bu(i)$ for $y(t) = C(zI-A)^{-1}B[u](t)$, and its modification: $w(t) = C\sum_{i=0}^{t}A^{t-1}Bu(i)$ for $w(t) = C(zI-A)^{-1}Bz[u](t) = C(zI-A)^{-1}B[u](t+1)$, we express

$$\begin{aligned}
&h_c(z)[(h_b(z)z)[\omega^T](z-1)[\theta]](t)\\
&= c\sum_{\tau=0}^{t-1}A^{t-\tau-1}\sum_{i=0}^{\tau}A^{\tau-i}b\,\omega^T(i)(\theta(\tau+1)-\theta(\tau))\\
&= c\sum_{i=0}^{t-1}A^{t-i-1}b\,\omega^T(i)\theta(t)\\
&\quad + c\sum_{\tau=0}^{t-2}A^{t-\tau-1}\sum_{i=0}^{\tau}A^{\tau-i}b\,\omega^T(i)\theta(\tau+1) - c\sum_{\tau=0}^{t-1}A^{t-\tau-1}\sum_{i=0}^{\tau}A^{\tau-i}b\,\omega^T(i)\theta(\tau)\\
&= \theta^T(t)h(z)[\omega](t) + c\sum_{\sigma=1}^{t-1}A^{t-\sigma}\sum_{i=0}^{\sigma-1}A^{\sigma-i-1}b\,\omega^T(i)\theta(\sigma)\\
&\quad - c\sum_{\tau=0}^{t-1}A^{t-\tau-1}b\,\omega^T(\tau)\theta(\tau) - c\sum_{\tau=1}^{t-1}A^{t-\tau-1}\sum_{i=0}^{\tau-1}A^{\tau-i}b\,\omega^T(i+1)\theta(\tau)\\
&= \theta^T(t)h(z)[\omega](t) - h(z)[\theta^T\omega](t),
\end{aligned} \tag{24}$$

where, by definition,

$$c\sum_{i=0}^{t-1}A^{t-i-1}b\,\omega(i) = h(z)[\omega](t) \tag{25}$$

$$c\sum_{\tau=0}^{t-1}A^{t-\tau-1}b\,\omega^T(\tau)\theta(\tau) = h(z)[\theta^T\omega](t). \tag{26}$$

This is the discrete-time version of the swapping lemma (5.331), whose explicit version can also be derived, similar to the continuous-time case above.

# Additional Lemmas

### A Small Gain Lemma

**Lemma 1** *For* $\|w\|_t = \sup_{0\le\tau\le t} |w(\tau)|$, *if*

$$|w(t)| \le \beta(t)\|w\|_t + \gamma(t) \tag{27}$$

*for some* $\beta(t)$ *such that* $\lim_{t\to\infty} \beta(t) = 0$ *and some* $\gamma(t) \in L^\infty$*, then* $w(t)$ *is bounded, that is,* $w(t) \in L^\infty$*. If, in addition,* $\lim_{t\to\infty} \gamma(t) = 0$*, then* $\lim_{t\to\infty} w(t) = 0$.

**Proof**: Assume that $w(t)$ is not bounded. Then there exists a subsequence $\{t_n\}$ such that $\lim_{t_n\to\infty} |w(t_n)| = \infty$ and $|w(t)| \le |w(t_n)|$ for $t \le t_n$, that is, $\|w\|_{t_n} \le |w(t_n)|$, and it follows that

$$|w(t_n)| \le \beta(t_n)\|w\|_{t_n} + \gamma(t_n) \le \beta(t_n)|w(t_n)| + \gamma(t_n). \tag{28}$$

Since $\lim_{t_n\to\infty} \beta(t_n) = 0$ and $\gamma(t) \in L^\infty$, the above inequality implies that $w(t_n)$ is bounded, a contradiction to the assumption that $\lim_{t_n\to\infty} w(t_n) = \infty$ or $w(t)$ is unbounded. Hence, $w(t)$ is bounded, and so is $\|w\|_t$.

If $\lim_{t\to\infty} \gamma(t) = 0$ also holds, then from (27), the boundedness of $w(t)$ and the condition that $\lim_{t\to\infty} \beta(t) = 0$, it follows that $\lim_{t\to\infty} |w(t)| = \lim_{t\to\infty}(\beta(t)\|w\|_t + \gamma(t)) = 0$. $\nabla$

This result can be generalized to the vector signal case.

**Lemma 2** *For* $\|w\|_t = \sup_{0\le\tau\le t} \|w(\tau)\|$ *with* $\|w(t)\|$ *being a vector norm of* $w(t) \in R^n$*, if*

$$\|w(t)\| \le \beta(t)\|w\|_t + \gamma(t) \tag{29}$$

*for some* $\beta(t)$ *such that* $\lim_{t\to\infty} \beta(t) = 0$ *and some* $\gamma(t) \in L^\infty$*, then* $w(t)$ *is bounded, that is,* $w(t) \in L^\infty$*. If, in addition,* $\lim_{t\to\infty} \gamma(t) = 0$*, then* $\lim_{t\to\infty} w(t) = 0$.

**A Signal Convergence Lemma**

**Lemma 3** *If* $\ddot{e}(t) \in L^\infty$ *and* $\lim_{t\to\infty} e(t) = 0$*, then* $\lim_{t\to\infty} \dot{e}(t) = 0$.

**Proof**: We need to show that for any given $\eta > 0$, there exists a $T > 0$ such that $|\dot{e}(t)| < \eta$ for any $t > T$.

The proof makes use of two related fictitious filters with a virtual parameter $a > 0$: $H(s) = \frac{1}{s+a}$ and $K(s) = \frac{a}{s+a}$ such that $1 = sH(s) + K(s)$. Operating this identity on $\dot{e}(t)$, we have

$$\dot{e}(t) = H(s)s[\dot{e}](t) + sK(s)[e](t) = H(s)[\ddot{e}](t) + sK(s)[e](t).$$

Under the condition that $\ddot{e}(t) \in L^\infty$, for any given $\eta > 0$, $H(s)[\ddot{e}](t)$ can be made smaller than $\frac{\eta}{2}$ by a choice of a sufficiently large and finite $a$. Under the condition that $\lim_{t\to\infty} e(t) = 0$, for the chosen value of $a$, there exists a $T > 0$ such that for any $t > T$, $|sK(s)[e](t)| < \frac{\eta}{2}$.

This derivation means that for any given $\eta > 0$, there exists a $T > 0$ such that $|\dot{e}(t)| < \eta$ for any $t > T$, which is equivalent to the convergence of $\dot{e}(t)$ to 0 as $t \to \infty$. $\nabla$

**Remark 1** There are three possible situations:

(i) The signal $\dot{e}(t)$ converges to a constant: such a constant must be 0, otherwise $e(t)$ does not converge to 0 (leading to a contradiction to $\lim_{t\to\infty} e(t) = 0$); (ii) $\dot{e}(t)$ does not converge when $\ddot{e}(t)$ is bounded: $e(t)$ does not converge either (leading to a contradiction to $\lim_{t\to\infty} e(t) = 0$); and (iii) $\dot{e}(t)$ does not converge when $\ddot{e}(t)$ is unbounded: $e(t)$ may converge to 0 (e.g., $e(t) = \frac{\sin t^2}{t+1}$, $\dot{e}(t) = -\frac{\sin t^2}{(t+1)^2} + \frac{2t\cos t^2}{t+1}$, $\ddot{e}(t) = \frac{2\sin t^2}{(t+1)^3} - \frac{2t\cos t^2}{(t+1)^2} + \frac{2\cos t^2}{(t+1)^2} - \frac{4t^2\sin t^2}{t+1}$) (but this is not the case, as $\ddot{e}(t) \in L^\infty$ by assumption).

Hence, if $\ddot{e}(t) \in L^\infty$ and $\lim_{t\to\infty} e(t) = 0$, then $\lim_{t\to\infty} \dot{e}(t) = 0$. □

**Remark 2** This result also follows from the Barbalat Lemma: If a scalar function $f(t)$ is uniformly continuous such that $\lim_{t\to\infty} \int_0^t f(\tau)\,d\tau$ exists and is finite, then e $\lim_{t\to\infty} f(t) = 0$. In this case, $f(t) = \dot{e}(t)$, $\lim_{t\to\infty} \int_0^t f(\tau)\,d\tau = \lim_{t\to\infty} \int_0^t \dot{e}(\tau)\,d\tau = \lim_{t\to\infty}(e(t) - e(0)) = -e(0)$ exists and is finite, and $f(t)$ is uniformly continuous, as $\dot{f}(t) = \ddot{e}(t)$ is bounded. □

**Lemma** (Barbalat): If a scalar function $f(t)$ is uniformly continuous such that $\lim_{t\to\infty} \int_0^t f(\tau)\,d\tau$ exists and is finite, then $\lim_{t\to\infty} f(t) = 0$.

$\diamond$ Convergence of $e(t)$ (which can be a tracking error signal $e(t) = y(t) - y_m(t)$): With $f(t) = e^2(t)$, we have that $\dot{f}(t) = 2e(t)\dot{e}(t)$ is bounded, i.e., $f(t)$ is uniformly continuous, so that, with $e(t) \in L^2$ (that is, $\lim_{t\to\infty} \int_0^t f(\tau)\,d\tau$ exists and is finite), $\lim_{t\to\infty} f(t) = 0$, that is, $\lim_{t\to\infty} e(t) = 0$.

**Proposition 1**: If $\lim_{t\to\infty} e(t) = e_0$ being a constant and $\ddot{e}(t)$ is bounded, then $\lim_{t\to\infty} \dot{e}(t) = 0$.

• Barbalat lemma based proof

$\diamond$ Convergence of $\dot{e}(t)$ with $\lim_{t\to\infty} e(t) = 0$: For $f(t) = \dot{e}(t)$, $\ddot{e}(t)$ being bounded means $f(t)$ being uniformly continuous, and $\lim_{t\to\infty} \int_0^t f(\tau)\,d\tau = e(\infty) - e(0) = -e(0)$ exists and is finite, so that $\lim_{t\to\infty} f(t) = 0$, that is, $\lim_{t\to\infty} \dot{e}(t) = 0$.

$\diamond$ Convergence of $\dot{e}(t)$ with $\lim_{t\to\infty} e(t) = e_0 \in R$: For $f(t) = \dot{e}(t)$, $\ddot{e}(t)$ being bounded means $f(t)$ being uniformly continuous, and $\lim_{t\to\infty} \int_0^t f(\tau)\,d\tau = e(\infty) - e(0) = e_0 - e(0)$ exists and is finite, so that $\lim_{t\to\infty} \dot{e}(t) = 0$.

• Direct proof

$\diamond$ Convergence of $\dot{e}(t)$ with $\lim_{t\to\infty} e(t) = 0$: For $H(s) = \frac{1}{s+a}$ and $K(s) = \frac{a}{s+a}$ such that $1 = sH(s) + K(s)$, we have

$$\dot{e}(t) = sH(s)[\dot{e}](t) + K(s)[\dot{e}](t) = H(s)[\ddot{e}](t) + sK(s)[e](t)$$

whose first term can be virtually made arbitrarily small by a virtually large $a > 0$ and the second term converges to zero, that is, for any given $\eta > 0$, there exists a $T > 0$ such that $|\dot{e}(t)| < \eta$ for any $t > T$. This implies that $\dot{e}(t)$ converges to zero.

$\diamond$ Convergence of $\dot{e}(t)$ with $\lim_{t\to\infty} e(t) = e_0$ for any constant $e_0$: For $f(t) = e(t) - e_0$, $\lim_{t\to\infty} f(t) = 0$, which, with $\ddot{f}(t) = \ddot{e}(t)$ bounded, implies that $\dot{f}(t) = \dot{e}(t)$ goes to zero.

**Proposition 2** (the general case): For any $j > 0$, if $\lim_{t\to\infty} e(t) = 0$ and $\frac{d^i e(t)}{dt^i}$ are bounded, $i = 2, \ldots, j+1$, then $\lim_{t\to\infty} \frac{d^j e(t)}{dt^j} = 0$.

$\diamond$ Proof based on Proposition 1

To use the induction method to prove this result, from Proposition 1, we first have that the result is true for $j = 1$, that is, $\lim_{t\to\infty} e(t) = 0$ and boundedness of $\frac{d^2 e(t)}{dt^2}$ imply that $\lim_{t\to\infty} \frac{de(t)}{dt} = 0$.

Suppose it is true for $j = k$, that is, $\lim_{t\to\infty} \frac{d^k e(t)}{dt^k} = 0$ (with $\lim_{t\to\infty} e(t) = 0$ and $\frac{d^i e(t)}{dt^i}$ bounded, $i = 2, \ldots, k+1$). Then, for $j = k+1$, with the conditions that $\lim_{t\to\infty} \frac{d^k e(t)}{dt^k} = 0$ (or, $\lim_{t\to\infty} f(t) = 0$ for $f(t) = \frac{d^k e(t)}{dt^k}$) and that $\frac{d^i e(t)}{dt^i}$ is bounded for $i = k+2$ (or, $\frac{d^2 f(t)}{dt^2}$ is bounded), we have from Proposition 1 (or the result of $j = 1$) that $\lim_{t\to\infty} \frac{df(t)}{dt} = 0$ (that is, $\lim_{t\to\infty} \frac{d^{k+1} e(t)}{dt^{k+1}} = 0$). This shows that the induction process goes through, that is, the proposition is true for any $j$.

# Plant Zeros and Controller Order for MRAC

Consider a LTI system

$$\dot{x} = Ax + Bu, \; y = Cx \tag{30}$$

where $A \in R^{n\times n}$, $B \in R^{n\times M}$, $C \in R^{M\times n}$, whose transfer matrix $G(s) = C(sI-A)^{-1}B$ is an $M \times M$ strictly proper rational matrix.

For SISO systems with $M = 1$, we have $G(s) = \frac{Z(s)}{P(s)}$ for some nominal polynomials $Z(s)$ and $P(s)$ with $\partial P(s) = n_p \leq n$ (if $P(s)$ is taken as $P(s) = \det[sI-A]$, then $n_p = n$). If we take $n$ as the system order for MRAC design, that is, the controller order is $n-1 = \partial\Lambda(s)$, then we need to assume that all zeros of $Z(s)$ (which are the system zeros) are stable, for a stable MRAC system. With such a full-order representation $P(s) = \det[sI-A]$, the condition that all zeros of $Z(s)$ are stable implies that the realization $(A,B,C)$ is stabilizable and detectable. The choice of the controller order $n-1$ has another interpretation in the SISO case: the output feedback controller structure is a reparametrization of a state feedback control law $u = k_1^{*T}x + k_2^*r$, using a reduced-order (with order $n-1$) state observer (see Section 4.4.2).

For SISO systems, if, due to pole-zero cancellations caused by a nonminimal realization $(A,B,C)$, the degree $n_p$ of $P(s)$ is less than $n$: $n_p < n$, and the controller order is $n_p - 1$, we need to assume that, in addition to all zeros of $Z(s)$ being stable, the pole-zero cancellations leading $G(s) = C(sI-A)^{-1}B$ to $\frac{Z(s)}{P(s)}$ should be also stable ones, that is, the nonminimal realization $(A,B,C)$ should be stabilizable and detectable. In this case, the system zeros include the zeros of $Z(s)$ and those canceled zeros in $G(s)$, while the transfer function zeros are those $s_z$ which make $\frac{Z(s_z)}{P(s_z)}$ zero and they can be a subset of the zeros of $Z(s)$, as there may be some more zero-pole cancellations between $Z(s)$ and $P(s)$ in $\frac{Z(s_z)}{P(s_z)}$.

In the SISO case, a state feedback control design $u(t) = K_1^T x(t) + K_2 r(t)$ requires that all zeros of $Z(s)$ are stable (in this case, $P(s) = \det[sI-A]$) which is necessary and sufficient for stable plant-model matching: $\det[sI-A-BK_1^{*T}] = P_m(s)Z(s)/z_m$ (not needed for just pole placement control). An output feedback control design is based on a nominal transfer function $\frac{Z(s)}{P(s)}$ with order $n_p = \partial P(s)$, and all zeros of $Z(s)$ need to be stable for stable plant-model matching based on $\frac{Z(s)}{P(s)}$. If $n \leq n_p$, a stable $Z(s)$ implies that $(A,B,C)$ is stabilizable and detectable. If $n > n_p$, then there are zero-pole cancellations in $C(sI-A)^{-1}B = \frac{Z(s)}{P(s)}$, and $(A,C)$ needs to be detectable, otherwise those undetectable modes (eigenvalues of $A$) cannot be stabilized (output feedback does not change unobservable modes); $(A,B)$ needs to be stabilizable, otherwise those unstabilizable modes (eigenvalues of $A$) cannot be stabilized (output or state feedback does not change uncontrollable modes).

For MIMO systems, there are also different ways to express the system transfer matrix $G(s) = C(sI-A)^{-1}B$ (we first assume that $G(s)$ has full rank $m$ for an $M \times M$ $G(s)$, that is, $\text{rank}[G(s)] < M$ only for a finite number of values of $s$).

The first (simple) form of $G(s)$ is

$$G(s) = C(sI-A)^{-1}B = \frac{N(s)}{d(s)},\ d(s) = \det[sI-A],\ \partial d(s) = n. \tag{31}$$

If we use a controller of order $n-1 = \partial\Lambda(s)$, we need to assume that all zeros of $\det[N(s)]$ are stable, which may be thought as the system zeros (note that the zeros of $G(s)$, as defined in the literature, may be only a subset of those zeros if $(A,B,C)$ is not a minimal realization). In fact, different from the SISO case, the controller order can be reduced to $n-M$ in the MIMO case, because a reduced-order observer has order $n-M$, so that the state feedback control law $u = K_1^{*T}x + K_2^* r$ can be reparametrized using the input and output signals, leading to an $(n-M)$th order output feedback controller.

Given that the observability index of a minimal realization of $G(s)$ is $\nu \le n - M + 1$, that is, $\nu - 1 \le n - M$, the order of the usual MRAC structure, which has been chosen as $\nu - 1$, is the minimal order to meet the desired plant-model matching. If for a system, the order of a reduced-order observer can be chosen as $\nu - 1$, the parametrization of the output feedback controller with order $\nu - 1$ can then be seen as a reparametrization of a state feedback controller, using input and output signals. If not, the controller parametrization may be considered as being one used for plant-model matching. In MRAC for multivariable systems, we call $\nu$ the observability index of the system transfer matrix $G(s)$, in the sense that $\nu$ is the observability index of a minimal realization of the system transfer matrix $G(s)$.

The second form of $G(s)$ is a left matrix-fraction description:

$$G(s) = C(sI-A)^{-1}B = \bar{P}_l^{-1}(s)\bar{Z}_l(s) \tag{32}$$

where $\bar{P}_l(s)$ and $\bar{Z}_l(s)$ are $M \times M$ polynomial matrices with $\bar{P}_l(s)$ row reduced (a polynomial matrix $\bar{P}_l(s)$ is row reduced if the elements in the $i$th row of $\bar{P}_l(s)$ have a largest degree $\bar{\nu}_i$ and the matrix $\Gamma_r = \lim_{s\to\infty} \text{diag}\{s^{-\bar{\nu}_1}, s^{-\bar{\nu}_2}, \ldots, s^{-\bar{\nu}_M}\}\bar{P}_l(D)$ is nonsingular), and the $i$th row degree of $\bar{Z}_l(s)$ is less than the $i$th row degree of $\bar{P}_l(s)$, $i = 1,2,\ldots,q$ (it is denoted that $\partial\bar{P}_l(s) = \bar{\nu}$, that is, $\bar{\nu} = \max\{\bar{\nu}_i\}$). A controller can be designed with $\partial\Lambda(s) = \bar{\nu} - 1$. In this case, for stable MRAC, we need to assume that the zeros of $\det[\bar{Z}_l(s)]$ are stable and also that $(A,B,C)$ is stabilizable and detectable (as some pole-zero cancellations may occur when obtaining the system model $\bar{P}_l^{-1}(s)\bar{Z}_l(s)$ from $(A,B,C)$).

Note that a left matrix-fraction description $G(s) = \bar{P}_l^{-1}(s)\bar{Z}_l(s)$, can always be made to have a row reduced $\bar{P}_l(s)$, by using elementary row operations represented by a unimodular matrix $M(s)$ (such a matrix is defined to have a non-zero constant determinant) on a non-reduced $\tilde{P}_l(s)$:

$$G(s) = \tilde{P}_l^{-1}(s)\tilde{Z}_l(s) = (M(s)\tilde{P}_l(s))^{-1}M(s)\tilde{Z}_l(s) = \bar{P}_l^{-1}(s)\bar{Z}_l(s), \qquad (33)$$

for $\bar{P}_l(s) = M(s)\tilde{P}_l(s)$ and $\bar{Z}_l(s) = M(s)\tilde{Z}_l(s)$.

The third form of $G(s)$ is a left co-prime matrix-fraction description:

$$G(s) = C(sI-A)^{-1}B = P_l^{-1}(s)Z_l(s) \qquad (34)$$

where $P_l(s)$ and $Z_l(s)$ are left co-prime $M \times M$ polynomial matrices ($P_l(s)$ and $Z_l(s)$ are left co-prime if any $M \times M$ polynomial matrix $W(s)$ such that $Z_l(s) = W(s)Z(s)$ and $P_l(s) = W(s)P(s)$ for some polynomial matrices $Z(s)$ and $P(s)$—such a $W(s)$ is called a common left divisor of $Z_l(s)$ and $P_l(s)$, is a unimodular matrix, that is, $\det[W(s)]$ is a non-zero constant), with $P_l(s)$ row reduced and $\partial P_l(s) = \nu \leq \bar{\nu}$. A controller can be designed with $\partial\Lambda(s) = \nu - 1$. In this case, we also need to assume that the zeros of $\det[Z_l(s)]$ are stable and that $(A,B,C)$ is stabilizable and detectable.

The situation is similar for $G(s)$ in a right matrix-fraction description:

$$G(s) = C(sI-A)^{-1}B = Z_r(s)P_r^{-1} \qquad (35)$$

with $P_r(s)$ being column reduced (that is, $P_r^T(s)$ being row reduced) and $Z_r(s)$ and $P_r(s)$ being right co-prime (that is, $Z_r^T(s)$ and $P_r^T(s)$ being left co-prime). In particular, the column degrees of $P_r(s)$ are denoted as $\mu_i$ (or $\bar{\mu}_i$ for $\bar{P}_r(s)$ if $G(s) = C(sI-A)^{-1}B = \bar{Z}_r(s)\bar{P}_r^{-1}$ with $\bar{Z}_r(s)$ and $\bar{P}_r(s)$ not right co-prime), $i = 1,2,\ldots,m$, and the column degrees of $Z_r(s)$ ($\bar{Z}_r(s)$) are less than that of $P_r(s)$ ($\bar{P}_r(s)$). If $(A,B,C)$ is a minimal realization, then $\mu_i$, $i = 1,2,\ldots,m$, are the controllability indexes of $(A,B)$. If $(A,B,C)$ is not a minimal realization, then one can a controllable realization $(\bar{A}_c,\bar{B}_c,\bar{C}_c)$ whose controllability indexes are $\bar{\mu}_i$, for $\bar{Z}_r(s)\bar{P}_r^{-1}(s)$ (or $(A_c,B_c,C_c)$ whose controllability indexes are $\mu_i$, for $Z_r(s)P_r^{-1}$).

Note that if $(A,B,C)$ is a minimal realization, then $\nu$ is the observability index of $(A,C)$ (we may also call $\nu$ the observability index of $G(s)$). When $(A,B,C)$ is not minimal, for the second case (or the third case), we can find an observable realization $(\bar{A}_o,\bar{B}_o,\bar{C}_o)$ whose observability indexes are $\bar{\nu}_i$, for $\bar{P}_l^{-1}(s)\bar{Z}_l(s)$ (or $(A_o,B_o,C_o)$ whose observability indexes are $\nu_i$, for $P_l^{-1}(s)Z_l(s)$). In this sense, we may also call $\bar{\nu}$ (or $\nu$) the observability index of $G(s) = \bar{P}_l^{-1}(s)\bar{Z}_l(s)$ (or $G(s) = P_l^{-1}(s)Z_l(s)$).

In summary, for MRAC of a system $(A,B,C)$, a basic assumption is that $(A,B,C)$ is stabilizable and detectable, in addition to the assumption that all zeros of the system transfer matrix $G(s) = C(sI-A)^{-1}B$ are stable; in other words, all system zeros are required to be stable.

For more about the definitions of the zeros and poles of $G(s)$, see Kailath (1980) and Rugh (1996), and also see more notes.

# Parameter Convergence of MRAC

## State Feedback State Tracking MRAC

Consider a linear time-invariant plant in the state-space form:

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(t) \in R^n,\ u(t) \in R, \tag{36}$$

where $A \in R^{n\times n}$, $B \in R^n$ are some unknown constant parameter matrix and parameter and vector, and assume that the state vector $x(t)$ is available for measurement.

The control objective is to design a state feedback control law for $u(t)$ such that all signals in the closed-loop system are bounded and the plant state vector $x(t)$ asymptotically tracks a reference state vector $x_m(t)$ generated from a chosen reference model system

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t),\ x_m(t) \in R^n,\ r(t) \in R \tag{37}$$

where $A_m \in R^{n\times n}$ is stable, and $B_m \in R^n$, both are constant.

To meet such a state tracking control objective, we assume that there exist a constant vector $k_1^* \in R^n$ and a nonzero constant scalar $k_2^* \in R$ to satisfy the plant-model matching conditions:

$$A + Bk_1^{*T} = A_m,\ Bk_2^* = B_m. \tag{38}$$

The adaptive controller structure is chosen as

$$u(t) = k_1^T(t)x(t) + k_2(t)r(t), \tag{39}$$

where $k_1(t)$ and $k_2(t)$ are the estimates of $k_1^*$ and $k_2^*$.

For the tracking error $x(t) - x_m(t)$ and the parameter errors

$$\tilde{k}_1(t) = k_1(t) - k_1^*,\ \tilde{k}_2(t) = k_2(t) - k_2^*, \tag{40}$$

the tracking error equation can be derived as

$$\dot{e}(t) = A_m e(t) + B\left(\tilde{k}_1^T(t)x(t) + \tilde{k}_2(t)r(t)\right). \tag{41}$$

The adaptive laws for $k_1(t)$ and $k_2(t)$ are chosen as

$$\dot{k}_1(t) = -\mathrm{sign}[k_2^*]\Gamma x(t)e^T(t)PB_m, \tag{42}$$

$$\dot{k}_2(t) = -\mathrm{sign}[k_2^*]\gamma r(t)e^T(t)PB_m, \tag{43}$$

where $\Gamma = \Gamma^T > 0$ and $\gamma > 0$, and $P \in R^{n\times n}$ such that $P = P^T > 0$ satisfying

$$PA_m + A_m^T P = -Q \tag{44}$$

for a chosen constant matrix $Q \in R^{n\times n}$ such that $Q = Q^T > 0$.

Since $\bar{\Gamma} = \text{diag}\{\Gamma, \gamma\} = \bar{\Gamma}^T > 0$, there exists a nonsingular matrix $\bar{\Gamma}_1$ such that $\bar{\Gamma} = \bar{\Gamma}_1^T \bar{\Gamma}_1$. With such a $\bar{\Gamma}_1$, we introduce the overall parameter error vector

$$\tilde{\bar{\theta}}(t) = (\bar{\Gamma}_1^T)^{-1}\tilde{\theta}(t),\ \tilde{\theta}(t) = [\tilde{k}_1^T(t), \tilde{k}_2(t)]^T \in R^{n+1}, \tag{45}$$

and the corresponding regressor vector

$$\bar{\omega}(t) = \bar{\Gamma}_1 \omega(t),\ \omega(t) = [x^T(t), r(t)]^T \in R^{n+1}. \tag{46}$$

Since $\bar{P} = k_2^* P = \bar{P}^T > 0$, there is a nonsingular matrix $\bar{P}_1 = \bar{P}_1^T \in R^{n\times n}$ such that $\bar{P} = \bar{P}_1 \bar{P}_1$.[1] With such a nonsingular $\bar{P}_1 = \bar{P}_1^T$, we introduce the transformed tracking error

$$\bar{e}(t) = \bar{P}_1 e(t). \tag{47}$$

Then, with $\bar{A}_m = \bar{P}_1 A_m \bar{P}_1^{-1}$ and $\bar{B} = \bar{P}_1 B$, we can express the tracking error equation (41) as

$$\dot{\bar{e}}(t) = \bar{P}_1 A_m e(t) + \bar{P}_1 B \omega^T(t) \tilde{\theta}(t) = \bar{A}_m \bar{e}(t) + \bar{B}\bar{\omega}^T(t)\tilde{\bar{\theta}}(t) \tag{48}$$

and, for $k_2^* > 0$, write the adaptive laws (42)–(43) as

$$\dot{\tilde{\bar{\theta}}}(t) = -\bar{\omega}(t)\bar{B}^T \bar{e}(t). \tag{49}$$

In a compact form, we have

$$\begin{bmatrix} \dot{\bar{e}}(t) \\ \dot{\tilde{\bar{\theta}}}(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_m & \bar{B}\bar{\omega}^T(t) \\ -\bar{\omega}(t)\bar{B}^T & 0 \end{bmatrix} \begin{bmatrix} \bar{e}(t) \\ \tilde{\bar{\theta}}(t) \end{bmatrix}. \tag{50}$$

Here, the matrix $\bar{A}_m = \bar{P}_1 A_m \bar{P}_1^{-1}$ with $\bar{P} = k_2^* P = \bar{P}_1 \bar{P}_1$ and $\bar{P}_1 = \bar{P}_1^T$ has the property:

$$\begin{aligned} \bar{A}_m + \bar{A}_m^T &= \bar{P}_1^{-1}(\bar{P}A_m + A_m^T \bar{P})\bar{P}_1^{-1} \\ &= k_2^* \bar{P}_1^{-1}(PA_m + A_m^T P)(\bar{P}_1^{-1})^T = -k_2^* \bar{P}_1^{-1} Q (\bar{P}_1^{-1})^T < 0. \end{aligned} \tag{51}$$

To study the convergence properties of the parameter error $\tilde{\theta}(t)$ in (41), or equivalently, that of $\tilde{\bar{\theta}}(t)$ in (50), we first introduce the following definition:

**Definition 1** *A bounded vector signal $x(t) \in R^q$, $q \geq 1$, is persistently exciting (PE) if there exist $\delta > 0$ and $\alpha_0 > 0$ such that*

$$\int_\sigma^{\sigma+\delta} x(t)x^T(t)\, dt \geq \alpha_0 I,\ \forall \sigma \geq t_0. \tag{52}$$

[1] In this case, there is a nonsingular matrix $\bar{Q} \in R^{n\times n}$ such that $\bar{Q}\bar{Q}^T = I$ and $\bar{P} = \bar{Q}\bar{\Lambda}\bar{Q}^T$ with $\bar{\Lambda}$ being diagonal whose diagonal elements are the eigenvalues of $\bar{P}$, which are all real and positive. Then, $\bar{P} = \bar{P}_1\bar{P}_1$, where $\bar{P}_1 = \bar{Q}\bar{\Lambda}_1\bar{Q}^T$ with $\bar{\Lambda}_1$ being diagonal such that $\bar{\Lambda}_1\bar{\Lambda}_1 = \bar{\Lambda}$, which leads to $\bar{P}_1 = \bar{P}_1^T > 0$ and $\bar{P}_1^{-1} = (\bar{P}_1^{-1})^T$.

We now present the following results.

**Result** (Morgan and Narendra 1977; Narendra and Annaswamy 1989): For the system

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\phi}(t) \end{bmatrix} = \begin{bmatrix} A_0 & B_0\zeta_0^T(t) \\ -\zeta_0(t)B_0^T & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ \phi(t) \end{bmatrix}, \tag{53}$$

with $z(t) \in R^n$, $\phi(t) \in R^{n+1}$, $\zeta_0(t) \in R^{n+1}$, $A_0 \in R^{n\times n}$ being stable (i.e., all its eigenvalues are in $\mathrm{Re}[s] < 0$) such that $A_0 + A_0^T < 0$, and $B_0 \in R^n$ and $(A_0, B_0)$ being controllable, if $\zeta_0(t)$ is bounded and PE, then $\lim_{t\to\infty} \phi(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$, both exponentially.

**Result**: The MRAC system has been transformed to (50) (with $\bar{A}_m = \bar{P}_1 A_m \bar{P}_1^{-1}$ stable and $\bar{A}_m + \bar{A}_m^T < 0$ as shown in (51), and $\bar{\omega}(t) = \bar{\Gamma}_1\omega(t)$) which has the same form as that in (53).

**Remark**: $\omega(t)$ is PE iff and $\bar{\omega}(t)$ is PE, for $\bar{\omega}(t) = \bar{\Gamma}_1\omega(t)$.

**Result** (Narendra and Annaswamy (1989); Ioannou and Sun (1996)): For the system

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t),\ x_m(t) \in R^n,\ r(t) \in R \tag{54}$$

where $A_m \in R^{n\times n}$ is stable and $(A_m, B_m)$ is controllable, if the input signal $r(t)$ has $n+1$ or more frequencies, then $\omega_m(t) = [x_m^T(t), r(t)]^T$ is PE.

**Remark**: $r(t) = 1$ has one frequency at 0, and $r(t) = \sin 2t$ has two frequencies at 2 and $-2$, etc.

**Result**: If $(\omega(t) - \omega_m(t)) \in L^2$ and $\omega(t)$ is PE, then $\omega_m(t)$ is PE.

This results can be proved by using the PE and $L^2$ signal definitions (conditions).

**Result**: MRAC ensures that all closed-loop system signals are bounded and $(\omega(t) - \omega_m(t)) \in L^2$.

**Remark**: $(A, B)$ is controllable iff $(A_m, B_m)$ is controllable iff $(\bar{A}_m, \bar{B})$ is controllable.

**Remark**: $\lim_{t\to\infty} \tilde{\theta}(t) = 0$ and $\lim_{t\to\infty} e(t) = 0$, both exponentially iff $\lim_{t\to\infty} \tilde{\bar{\theta}}(t) = 0$ and $\lim_{t\to\infty} \bar{e}(t) = 0$, both exponentially, for $\tilde{\bar{\theta}}(t) = (\bar{\Gamma}_1^T)^{-1}\tilde{\theta}(t)$ and $\bar{e}(t) = \bar{P}_1 e(t)$.

In conclusion, with the application of the result for (53) to the MRAC system equation (50), the state feedback state tracking MRAC system, with $(A, B)$ controllable and $r(t)$ of $n+1$ or more frequencies, ensures that $\lim_{t\to\infty} \tilde{\theta}(t) = 0$ and $\lim_{t\to\infty} e(t) = 0$, both exponentially.

## Output Feedback Output Tracking MRAC

Consider a linear time-invariant plant of the form

$$y(t) = G(s)[u](t),\ G(s) = k_p \frac{Z(s)}{P(s)} \tag{55}$$

with some unknown monic polynomials $P(s)$ and $Z(s)$ of degrees $n$ and $m$ respectively, and some gain $k_p$ known or unknown, and assume that $Z(s)$ is a stable polynomial for MRAC.

### MRAC System with $k_p$ Known

For the case of $k_p$ known, the adaptive controller structure is

$$u(t) = \theta_1^T \omega_1(t) + \theta_2^T \omega_2(t) + \theta_{20} y(t) + \theta_3^* r(t), \tag{56}$$

where

$$\omega_1(t) = \frac{a(s)}{\Lambda(s)}[u](t),\ \omega_2(t) = \frac{a(s)}{\Lambda(s)}[y](t), \tag{57}$$

$$a(s) = [1, s, \ldots, s^{n-2}]^T, \tag{58}$$

$\theta_1 \in R^{n-1}$, $\theta_2 \in R^{n-1}$ and $\theta_{20} \in R$ are parameters to be adaptively updated, $\Lambda(s)$ is a monic stable polynomial of degree $n-1$, and $\theta_3^* = k_p^{-1}$.

The control objective is to design the input signal $u(t)$ to ensure all closed-loop signals bounded and the output $y(t)$ asymptotically tracking the reference output signal $y_m(t)$ satisfying

$$P_m(s)[y_m](t) = r(t), \tag{59}$$

where $P_m(s)$ is a monic stable polynomial of degree $n^* = n - m$, and $r(t)$ is a bounded and piecewise continuous reference input signal.

The parameters $\theta_1 \in R^{n-1}$, $\theta_2 \in R^{n-1}$ and $\theta_{20} \in R$ are estimates of the nominal parameters $\theta_1^* \in R^{n-1}$, $\theta_2^* \in R^{n-1}$ and $\theta_{20}^* \in R$ satisfying the matching equation

$$\theta_1^{*T} a(s)P(s) + (\theta_2^{*T} a(s) + \theta_{20}^* \Lambda(s))k_p Z(s) = \Lambda(s)(P(s) - k_p \theta_3^* Z(s) P_m(s)). \tag{60}$$

In this case, the regressor vector is defined as

$$\omega_0(t) = [\omega_1^T(t), \omega_2^T(t), y(t)]^T \in R^{2n-1}, \tag{61}$$

the corresponding filtered regressor vector is

$$\zeta_0(t) = W_m(s)[\omega_0](t),\ W_m(s) = \frac{1}{P_m(s)} \tag{62}$$

for a stable polynomial $P_m(s)$ of degree $n^*$, the estimation error is defined as

$$\epsilon(t) = e(t) + k_p \xi(t), \tag{63}$$

where $e(t) = y(t) - y_m(t)$, and

$$\xi(t) = \theta_0^T(t)\zeta_0(t) - \frac{1}{P_m(s)}[\theta_0^T \omega_0](t) \tag{64}$$

$$\theta_0(t) = [\theta_1^T(t), \theta_2^T(t), \theta_{20}(t)]^T \in R^{2n-1}. \tag{65}$$

The adaptive law for $\theta_0(t)$ is chosen as

$$\dot{\theta}_0(t) = -\text{sign}[k_p]\Gamma \frac{\zeta_0(t)\epsilon(t)}{m^2(t)} \tag{66}$$

where $\Gamma = \Gamma^T > 0$, and

$$m(t) = \sqrt{1 + \zeta_0^T(t)\zeta_0(t) + \xi^2(t)}. \tag{67}$$

The tracking error $e(t)$ satisfies

$$e(t) = \frac{k_p}{P_m(s)}[\tilde{\theta}_0^T \omega_0](t), \tag{68}$$

where $\tilde{\theta}_0(t) = \theta_0(t) - \theta_0^*$, with $\theta_0^* = [\theta_1^{*T}, \theta_2^{*T}, \theta_{20}^*]^T \in R^{2n-1}$ being the nominal value of $\theta_0(t)$ and depends on the parameters of $P(s)$ and $Z(s)$ as the plant-model matching parameter vector. The estimation error $\epsilon(t)$ can be expressed as

$$\epsilon(t) = \tilde{\theta}_0^T(t)\zeta_0(t), \tag{69}$$

and the adaptive law with $k_p > 0$ can be expressed as

$$\dot{\tilde{\theta}}_0(t) = -\Gamma \frac{\zeta_0(t)\zeta_0^T(t)\tilde{\theta}_0(t)}{m^2(t)}. \tag{70}$$

Since $\Gamma = \Gamma^T > 0$, there exists a nonsingular matrix $\Gamma_1$ such that $\Gamma = \Gamma_1^T \Gamma_1$. Hence, we have

$$(\Gamma_1^T)^{-1}\dot{\tilde{\theta}}_0(t) = -\frac{\Gamma_1 \zeta_0(t)\zeta_0^T(t)\Gamma_1^T}{m^2(t)}(\Gamma_1^T)^{-1}\tilde{\theta}_0(t). \tag{71}$$

Letting $\phi_0(t) = (\Gamma_1^T)^{-1}\tilde{\theta}_0(t)$ and $\psi_0(t) = \frac{\Gamma_1\zeta_0(t)}{m(t)}$, we arrive at

$$\dot{\phi}_0(t) = -\psi_0(t)\psi_0^T(t)\phi_0(t). \tag{72}$$

MRAC ensures the desired system properties: the closed-loop signal boundedness (which implies that the above $\psi_0(t)$ is bounded), the asymptotic output tracking: $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$, and the $L^2$ tracking error $e(t) = y(t) - y_m(t)$: $\int_0^\infty e^2(t)dt < \infty$. To study the convergence properties of the parameter error $\tilde{\theta}_0(t)$, or that of $\phi_0(t)$, we recall the following definition:

**Definition 2** *A bounded vector signal $x(t) \in R^q$, $q \geq 1$, is persistently exciting (PE) if there exist $\delta > 0$ and $\alpha_0 > 0$ such that*

$$\int_{\sigma}^{\sigma+\delta} x(t)x^T(t)\,dt \geq \alpha_0 I,\ \forall \sigma \geq t_0. \tag{73}$$

From the definition of $\zeta_0(t)$ and $u(t) = G^{-1}(s)[y](t)$, we have

$$\begin{aligned}\zeta_0(t) &= W_m(s)[\frac{a^T(s)}{\Lambda(s)}[u](t), \frac{a^T(s)}{\Lambda(s)}[y](t), y(t)]^T \\ &= W_m(s)[\frac{a^T(s)}{\Lambda(s)}G^{-1}(s)[y](t), \frac{a^T(s)}{\Lambda(s)}[y](t), y(t)]^T, \end{aligned} \tag{74}$$

where $W_m(s) = \frac{1}{P_m(s)}$, In a similar structure, we introduce the signal

$$\begin{aligned}\zeta_{0m}(t) &= W_m(s)[\frac{a^T(s)}{\Lambda(s)}G^{-1}(s)[y_m](t), \frac{a^T(s)}{\Lambda(s)}[y_m](t), y_m(t)]^T \\ &= W_m(s)H_0(s)[r](t) \in R^{2n-1}, \end{aligned} \tag{75}$$

where

$$H_0(s) = [\frac{a^T(s)}{\Lambda(s)}G^{-1}(s)W_m(s), \frac{a^T(s)}{\Lambda(s)}W_m(s), W_m(s)]^T. \tag{76}$$

The following results are available in the literature.

**Result** (Anderson 1997): If $\psi_0(t)$ in (72) is PE, then $\lim_{t\to\infty} \phi_0(t) = 0$ exponentially, that is, $\lim_{t\to\infty}(\theta_0(t) - \theta_0^*) = 0$ exponentially.

**Result**: If the signal $\zeta_0(t)$ is PE, then $\psi_0(t)$ is PE (which follows from the boundedness of $m(t)$).

**Result**: MRAC ensures that $(\zeta_0(t) - \zeta_{0m}(t)) \in L^2$ (which can be seen from $\zeta_0(t) - \zeta_{0m}(t) = H(s)[e](t)$, where $H(s)$ is strictly proper and stable, and $e(t) \in L^2$).

**Result**: Given that $(\zeta_0(t) - \zeta_{0m}(t)) \in L^2$, $\zeta_0(t)$ is PE iff $\zeta_{0m}(t)$ is PE.

**Result** (Boyd and Satry 1983): If $r(t)$ has $2n-1$ or more frequencies and $P(s)$ and $Z(s)$ are coprime, then $\zeta_{0m}(t)$ is PE.

In conclusion, for a MRAC system with $P(s)$ and $Z(s)$ coprime and $k_p$ known, if the reference signal $r(t)$ has $2n-1$ or more frequencies, then $\lim_{t\to\infty}(\theta(t) - \theta_0^*) = 0$ and $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$, both exponentially.

**MRAC System with $k_p$ Unknown**

In this case, the adaptive controller structure is

$$u(t) = \theta_1^T \omega_1(t) + \theta_2^T \omega_2(t) + \theta_{20} y(t) + \theta_3 r(t), \tag{77}$$

the regressor vector is defined as

$$\omega(t) = [\omega_1^T(t), \omega_2^T(t), y(t), r(t)]^T \in R^{2n}, \tag{78}$$

the corresponding filtered regressor vector is

$$\zeta(t) = W_m(s)[\omega](t),\ W_m(s) = \frac{1}{P_m(s)}, \tag{79}$$

the estimation error is defined as

$$\varepsilon(t) = e(t) + \rho(t)\xi(t), \tag{80}$$

where $\rho(t)$ is the estimate of $k_p$, and

$$\xi(t) = \theta^T(t)\zeta(t) - \frac{1}{P_m(s)}[\theta^T \omega](t) \tag{81}$$

$$\theta(t) = [\theta_1^T(t), \theta_2^T(t), \theta_{20}(t), \theta_3(t)]^T \in R^{2n}, \tag{82}$$

and the adaptive laws for $\theta(t)$ and $\rho(t)$ are chosen as

$$\dot{\theta}(t) = -\text{sign}[k_p]\Gamma \frac{\zeta(t)\varepsilon(t)}{m^2(t)} \tag{83}$$

$$\dot{\rho}(t) = -\frac{\gamma\varepsilon(t)\xi(t)}{m^2(t)} \tag{84}$$

where $\Gamma = \Gamma^T > 0$, $\gamma > 0$, and and

$$m(t) = \sqrt{1 + \zeta^T(t)\zeta(t) + \xi^2(t)}. \tag{85}$$

In this case, the tracking error $e(t)$ satisfies

$$e(t) = \frac{k_p}{P_m(s)}[\tilde{\theta}^T \omega](t), \tag{86}$$

where $\tilde{\theta}(t) = \theta(t) - \theta^*$, with $\theta^* = [\theta_1^{*T}, \theta_2^{*T}, \theta_{20}^*, \theta_3^*]^T \in R^{2n}$ being the nominal parameter vector. The estimation error $\varepsilon(t)$ can be expressed as

$$\varepsilon(t) = \rho^* \tilde{\theta}^T(t)\zeta(t) + \tilde{\rho}(t)\xi(t), \tag{87}$$

where $\rho^* = k_p$, $\tilde{\theta}(t) = \theta(t) - \theta^*$ and $\tilde{\rho}(t) = \rho(t) - \rho^*$. Introducing $\tilde{\bar{\theta}}(t) = \bar{\theta}(t) - \bar{\theta}^*$ with

$$\bar{\theta}(t) = [\theta^T(t), \rho(t)]^T \in R^{2n+1},\ \bar{\theta}^* = [\theta^{*T}, \rho^*]^T \in R^{2n+1} \tag{88}$$

and defining $\bar{\Gamma} = \mathrm{diag}\{\Gamma k_p^{-1}, \gamma\}$ (for $k_p > 0$), and

$$\bar{\zeta}(t) = [k_p \zeta^T(t), \xi(t)]^T \in R^{2n+1} \tag{89}$$

Then, the adaptive laws with $k_p > 0$ can be expressed as

$$\dot{\tilde{\bar{\theta}}}(t) = -\bar{\Gamma} \frac{\bar{\zeta}(t)\bar{\zeta}^T(t)\tilde{\bar{\theta}}(t)}{m^2(t)}. \tag{90}$$

While this expression has the same form as that in (70), it does not render the same procedure for the convergence analysis of $\tilde{\bar{\theta}}(t)$, because $\bar{\zeta}(t)$ cannot be PE as $\lim_{t\to\infty} \xi(t) = 0$ from MRAC.

In this case, we consider (83) and (87) with $k_p > 0$:

$$\dot{\tilde{\theta}}(t) = -\Gamma k_p \frac{\zeta(t)\zeta^T(t)}{m^2(t)} \tilde{\theta}(t) - \Gamma \frac{\zeta(t)\tilde{\rho}(t)\xi(t)}{m^2(t)}, \tag{91}$$

express the signal $\zeta(t)$ as

$$\begin{aligned}\zeta(t) &= W_m(s)[\frac{a^T(s)}{\Lambda(s)}[u](t), \frac{a^T(s)}{\Lambda(s)}[y](t), y(t), r(t)]^T \\ &= W_m(s)[\frac{a^T(s)}{\Lambda(s)} G^{-1}(s)[y](t), \frac{a^T(s)}{\Lambda(s)}[y](t), y(t), r(t)]^T,\end{aligned} \tag{92}$$

and introduce the signal

$$\begin{aligned}\zeta_m(t) &= W_m(s)[\frac{a^T(s)}{\Lambda(s)} G^{-1}(s)[y_m](t), \frac{a^T(s)}{\Lambda(s)}[y_m](t), y_m(t), r(t)]^T \\ &= W_m(s)H(s)[r](t) \in R^{2n},\end{aligned} \tag{93}$$

where

$$H(s) = [\frac{a^T(s)}{\Lambda(s)} G^{-1}(s) W_m(s), \frac{a^T(s)}{\Lambda(s)} W_m(s), W_m(s), 1]^T. \tag{94}$$

We have the following results.

**Result**: If $r(t)$ has $2n$ or more frequencies, and $P(s)$ and $Z(s)$ are coprime, then $\zeta_m(t)$ is PE.

**Result**: MRAC ensures that $(\zeta(t) - \zeta_m(t)) \in L^2$, and $\xi(t) \in L^2$ and $\lim_{t\to\infty} \xi(t) = 0$.

**Result**: Given that $(\zeta(t) - \zeta_m(t)) \in L^2$, $\zeta(t)$ is PE iff $\zeta_m(t)$ is PE.

**Result**: If $\zeta(t)$ is PE, then the solution $\tilde{\theta}(t)$ of the homogeneous part of (91): $\dot{\tilde{\theta}}(t) = -\Gamma k_p \frac{\zeta(t)\zeta^T(t)}{m^2(t)} \tilde{\theta}(t)$, has the property: $\lim_{t\to\infty} \tilde{\theta}(t) = 0$ exponentially.

**Result**: If $\zeta(t)$ is PE, then the solution $\tilde{\theta}(t)$ of (91) has the property:

$$\lim_{t\to\infty} \tilde{\theta}(t) = 0,\ \tilde{\theta}(t) \in L^2. \tag{95}$$

This follows from the system signal boundedness and the $\zeta(t)$ PE condition, which imply that

$$\dot{\tilde{\theta}}(t) = -\Gamma k_p \frac{\zeta(t)\zeta^T(t)}{m^2(t)} \tilde{\theta}(t) \tag{96}$$

is an exponentially stable system, and from the properties: $\lim_{t\to\infty} \xi(t) = 0$ and $\xi(t) \in L^2$.

In conclusion, for a MRAC system with $P(s)$ and $Z(s)$ coprime and $k_p > 0$ unknown, if the reference signal $r(t)$ has $2n$ or more frequencies, then $\lim_{t\to\infty}(\theta(t) - \theta^*) = 0$ and $(\theta(t) - \theta^*) \in L^2$, in addition to the usual MRAC tracking error properties: $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$ and $(y(t) - y_m(t)) \in L^2$.

# Higher-Order Tracking Performance of MRAC

It has been shown that for a linear time-invariant plant $y(t) = G(s)[u](t)$, where $G(s) = k_p \frac{Z(s)}{P(s)}$ with all zeros of $Z(s)$ stable and with relative degree $n^* =$ degree of $P(s)-$ degree of $Z(s)$, a model reference adaptive controller to generate the plant input $u(t)$ ensures that the tracking error $e(t) = y(t) - y_m(t)$ has the desired asymptotic convergence property: $\lim_{t\to\infty} e(t) = 0$, where $y_m(t) = W_m(s)[r](t)$ is the reference output to be tracked by the plant output $y(t)$, for a bounded reference input signal $r(t)$ and a stable transfer function $W_m(s) = \frac{1}{P_m(s)}$ with a stable polynomial $P_m(s)$ of degree $n^*$. In this work, we show that the tracking error $e(t)$ in a model reference adaptive control (MRAC) system has some stronger convergence properties as stated in the following theorem.

**Theorem 1** *For a model reference adaptive control system with the plant relative degree $n^* > 0$, the tracking error $e(t) = y(t) - y_m(t)$ has the convergence property:* $\lim_{t\to\infty} \frac{d^i e(t)}{dt^i} = 0$*, for $i = 0, 1, \ldots, n^* - 1$.*

**Proof**: The property: $\lim_{t\to\infty} e(t) = 0$ follows directly from the established properties: $\int_0^\infty e^2(t)dt < \infty$ and $\frac{de(t)}{dt}$ is bounded. However, it is not clear whether $\frac{d^i e(t)}{dt^i}$ is square-integrable or not for $i > 0$. To prove the convergence of $\frac{d^i e(t)}{dt^i}$, for $i = 1, \ldots, n^* - 1$, we recall that for a function $f(t)$ defined on $[t_0, \infty)$, $\lim_{t\to\infty} f(t) = 0$ if for every $\eta > 0$, there exists a $T = T(\eta) > 0$ such that $|f(t)| < \eta$, $\forall t > T$. Hence, our goal now is to show that in each case of $i = 1, \ldots, n^* - 1$, for any given $\eta$, there exists a $T > 0$ such that $|\frac{d^i e(t)}{dt^i}| < \eta$, for all $t > T$. To reach this goal, we use a method to decompose the signal $\frac{d^i e(t)}{dt^i}$ into two fictitious parts: one being small enough and one converging to zero asymptotically with time going to infinity. To this end, we first introduce two fictitious filters $K(s)$ and $H(s)$ from

$$K(s) = \frac{a^{n^*}}{(s+a)^{n^*}},\ sH(s) = 1 - K(s) \tag{97}$$

where $a > 0$ is a generic constant to be specified. The filter $H(s)$ is given as

$$H(s) = \frac{1}{s}(1 - K(s)) = \frac{1}{s} \frac{(s+a)^{n^*} - a^{n^*}}{(s+a)^{n^*}} \tag{98}$$

which is strictly proper (with relative degree 1) and stable and whose impulse response function is

$$h(t) = \mathcal{L}^{-1}[H(s)] = e^{-at} \sum_{i=1}^{n^*} \frac{a^{n^*-i}}{(n^*-i)!} t^{n^*-i}. \tag{99}$$

It can be verified that the $L^1$ signal norm of $h(t)$ is

$$\|h(\cdot)\|_1 = \int_0^\infty |h(t)|\,dt = \frac{n^*}{a}. \tag{100}$$

It has been established that the tracking error $e(t) = y(t) - y_m(t)$ satisfies

$$e(t) = \frac{k_p}{P_m(s)}[\tilde{\theta}^T \omega](t), \tag{101}$$

where $P_m(s)$ is a stable polynomial of degree $n^*$, and $\tilde{\theta}(t)$ is the parameter error vector and $\omega(t)$ is the controller regressor vector, and both are bounded. Using (97): $1 = sH(s) + K(s)$, we express $\dot{e}(t) = \frac{de(t)}{dt}$ as

$$\begin{aligned}
\dot{e}(t) &= \frac{k_p s}{P_m(s)}[\tilde{\theta}^T \omega](t) \\
&= H(s)\frac{k_p s^2}{P_m(s)}[\tilde{\theta}^T \omega](t) + sK(s)\frac{k_p}{P_m(s)}[\tilde{\theta}^T \omega](t) \\
&= H(s)\frac{k_p s^2}{P_m(s)}[\tilde{\theta}^T \omega](t) + sK(s)[e](t).
\end{aligned} \tag{102}$$

To demonstrate the new technique for proving the new result in Theorem 1: $\lim_{t\to\infty} \frac{d^i e(t)}{dt^i} = 0$, for $i = 1, \ldots, n^* - 1$, under the previously proved properties of MRAC: all closed-loop signals are bounded and the tracking error $e(t) = y(t) - y_m(t)$ satisfies $\lim_{t\to\infty} e(t) = 0$, we first consider the case of $n^* = 2$, where

$$K(s) = \frac{a^2}{(s+a)^2},\ H(s) = \frac{s+2a}{(s+a)^2}. \tag{103}$$

Since $\tilde{\theta}^T(t)\omega(t)$ is bounded and $\frac{k_p s^2}{P_m(s)}$ is stable and proper, $\frac{k_p s^2}{P_m(s)}[\tilde{\theta}^T \omega](t)$ is bounded. It follows from the above $L^1$ signal norm expression of $H(s)$ that

$$|H(s)\frac{k_p s^2}{P_m(s)}[\tilde{\theta}^T \omega](t)| \leq \frac{c_0}{a} \tag{104}$$

for any $t \geq 0$ and some constant $c_0 > 0$ independent of $a > 0$. Since $\lim_{t\to\infty} e(t) = 0$ as established and $sK(s)$ is stable and strictly proper (with relative degree $n^* - 1 = 1$ in this case of $n^* = 2$), it follows, for any $a > 0$ in $K(s)$, that

$$\lim_{t\to\infty} sK(s)[e](t) = 0.^2 \tag{105}$$

[2] For a stable and strictly proper transfer function $W(s)$ and a system $z(t) = W(s)[v](t)$, if $\lim_{t\to\infty} v(t) = 0$, then $\lim_{t\to\infty} z(t) = 0$ (see p. 68 of Narandra and Annaswamy (1989) and and p. 263 of Tao (2003).

To show that, for every $\eta > 0$, there exists a $T > 0$ such that $|\dot{e}(t)| < \eta$, $\forall t > T$, we set $a > a(\eta) = \frac{2c_0}{\eta}$ for the fictitious filter $H(s)$ in (98) so that $\frac{c_0}{a} < \frac{\eta}{2}$ in (104), and let $T = T_a(a(\eta), \eta) \stackrel{\triangle}{=} T(\eta) > 0$ such that $|sK(s)[e](t)| < \frac{\eta}{2}$ for all $t \geq T$ (which is ensured by the property that $\lim_{t\to\infty} sK(s)[e](t) = 0$ for any finite $a > 0$ in the other related fictitious filter $K(s)$ in (97)) (the peak value of $|sK(s)[e](t)|$ depends on the parameter $a$, so that the above time instant $T = T_a(a(\eta), \eta)$ also depends on $a$ too). Then, it follows from (102) and (104) that

$$|\dot{e}(t)| < \frac{\eta}{2} + \frac{\eta}{2} = \eta, \ \forall t > T, \tag{106}$$

which implies that $\lim_{t\to\infty} \dot{e}(t) = 0$. In other words, if $\lim_{t\to\infty} \dot{e}(t) \neq 0$, then there exist an $\eta_0 > 0$ and a sequence of time instants $t_i$ with $\lim_{i\to\infty} t_i = \infty$ such that $|\dot{e}(t_i)| > \eta_0$ for all $i = 1, 2, \ldots$, which, from the above analysis, is impossible.

Similarly, for the case when $n^* = 3$ with

$$K(s) = \frac{a^3}{(s+a)^3}, \ H(s) = \frac{s^2 + 3as + 3a^2}{(s+a)^3}, \tag{107}$$

we see that in (102), $\frac{k_p s^2}{P_m(s)}$ is stable and strictly proper, and so is $sK(s)$, so that $\lim_{t\to\infty} \dot{e}(t) = 0$. To show $\lim_{t\to\infty} \frac{d^2 e(t)}{dt^2} = 0$, from (102), we express $\frac{d^2 e(t)}{dt^2}$ as

$$\frac{d^2 e(t)}{dt^2} = H(s) \frac{k_p s^3}{P_m(s)} [\tilde{\theta}^T \omega](t) + s^2 K(s)[e](t), \tag{108}$$

in which $\frac{k_p s^3}{P_m(s)}[\tilde{\theta}^T \omega](t)$ is bounded (as $\frac{s^3}{P_m(s)}$ is stable and proper and $\tilde{\theta}^T(t)\omega(t)$ is bounded) and $H(s)$ satisfies (100), and $\lim_{t\to\infty} s^2 K(s)[e](t) = 0$. Hence, similar to (106), now with $\frac{d^2 e(t)}{dt^2}$ replacing $\dot{e}(t)$ in (106), it also follows that $\lim_{t\to\infty} \frac{d^2 e(t)}{dt^2} = 0$.

In general, for a MRAC system with an arbitrary relative degree $n^* > 0$, from (102), we express the $i$th-order time derivative $\frac{d^i e(t)}{dt^i}$ of $e(t)$ as

$$\frac{d^i e(t)}{dt^i} = H(s) \frac{k_p s^{i+1}}{P_m(s)} [\tilde{\theta}^T \omega](t) + s^i K(s)[e](t). \tag{109}$$

Since $\tilde{\theta}^T(t)\omega(t)$ is bounded, and for each $i = 1, 2, \ldots, n^* - 1$, $\frac{k_p s^{i+1}}{P_m(s)}$ is stable and strictly proper (proper for $i = n^* - 1$), we have

$$|H(s) \frac{k_p s^{i+1}}{P_m(s)} [\tilde{\theta}^T \omega](t)| \leq \frac{c_i}{a} \tag{110}$$

for some $c_i > 0$ independent of $a$. For each $i = 1, 2, \ldots, n^* - 1$, $s^i K(s)$ is stable and strictly proper, so that, with $\lim_{t\to\infty} e(t) = 0$, we have

$$\lim_{t\to\infty} s^i K(s)[e](t) = 0. \tag{111}$$

Hence, with the use of the fictitious parameter $a > 0$ in $H(s)$ and $K(s)$, similar to (106), it can be shown that for every $\eta > 0$, there exists a $T = T(\eta) > 0$ such that $|\frac{d^i e(t)}{dt^i}| < \eta$ for all $t > T$, so that $\lim_{t\to\infty} \frac{d^i e(t)}{dt^i} = 0$, for $i = 1, 2, \ldots, n^* - 1$. $\nabla$

For a general MRAC system, the reference input signal $r(t)$ is only required to be bounded (which is sufficient for the result of Theorem 1). If the time-derivative of $r(t)$ is also bounded, we have the following additional property.

**Corollary 1** *For a MRAC system with relative degree $n^* > 0$, if both $r(t)$ and $\dot{r}(t)$ are bounded, then the tracking error $e(t) = y(t) - y_m(t)$ has the convergence property: $\lim_{t\to\infty} \frac{d^i e(t)}{dt^i} = 0$, for $i = 0, 1, \ldots, n^*$.*

**Proof**: We just need to show the additional convergence property: $\lim_{t\to\infty} \frac{d^{n^*} e(t)}{dt^{n^*}} = 0$. To this end, we consider (109) with $i = n^*$:

$$\frac{d^{n^*} e(t)}{dt^{n^*}} = H(s) \frac{k_p s^{n^*}}{P_m(s)} s[\tilde{\theta}^T \omega](t) + s^{n^*} K(s)[e](t). \tag{112}$$

In this case, $s[\tilde{\theta}^T \omega](t) = \frac{d}{dt}(\tilde{\theta}^T(t)\omega(t)) = \dot{\tilde{\theta}}^T(t)\omega(t) + \tilde{\theta}^T \dot{\omega}(t)$ is bounded, because $\dot{\tilde{\theta}}(t)$ and $\dot{\omega}(t)$ (whose last component is $\dot{r}(t)$ which is bounded by assumption) are bounded. Hence $\frac{k_p s^{n^*}}{P_m(s)} s[\tilde{\theta}^T \omega](t)$ is bounded. Since $s^{n^*} K(s)$ in (112) is stable and proper and $H(s)$ satisfies (100), we have, with the property: $\lim_{t\to\infty} e(t) = 0$ and an inequality similar to (106) with $\frac{d^{n^*} e(t)}{dt^{n^*}}$ replacing $\dot{e}(t)$, that $\lim_{t\to\infty} \frac{d^{n^*} e(t)}{dt^{n^*}} = 0$. $\nabla$

**Remark 3** As a comparison, in the ideal case of nominal model reference control when $\theta(t) = \theta^*$ in the controller structure, it can be shown that

$$P_m(s)[y - y_m](t) = -k_p \epsilon_1(t) \tag{113}$$

for some exponentially decaying and initial condition related term $\epsilon_1(t)$, where $P_m(s)$ is a stable polynomial of degree $n^*$. It follows from this equation that the tracking error $e(t) = y(t) - y_m(t)$ has the convergence property: $\lim_{t\to\infty} \frac{d^i e(t)}{dt^i} = 0$, for $i = 0, 1, \ldots, n^*$ (under the condition that $r(t)$ is bounded). The difference between the nominal control and adaptive control cases is that in the adaptive control case the additional condition that $\dot{r}(t)$ is bounded was used to show $\lim_{t\to\infty} \frac{d^{n^*} e(t)}{dt^{n^*}} = 0$, which is not needed for the nominal control case. $\square$

**Remark 4** For the case when $y_m(t) = 0$, it follows that $\lim_{t\to\infty} \frac{d^i y(t)}{dt^i} = 0$, for $i = 0, 1, \ldots, n^* - 1$, an output regulation result of MRAC.

For a state-space system model: $\dot{x} = Ax + Bu, y = Cx$ with $x(t) \in R^n$, if $n^* = n$, then $\lim_{t\to\infty} \frac{d^i y(t)}{dt^i} = 0$, for $i = 0, 1, \ldots, n-1$, as established above, and $\lim_{t\to\infty} x(t) = 0$, as $(A, C)$ is observable implied by the condition that $n^* = n$ (the system transfer function does not have any finite zeros as its numerator is just a constant $k_p$). This is the adaptive asymptotic state regulation result, guaranteed by a MRAC design.

In particular, for an $n$th order system with $(A, B)$ in the controllable canonical form and $y(t) = x_1(t)$, the system transfer function explicitly has no finite zeros as its numerator is a constant, leading to $n^* = n$ and an observable $(A, C)$, so that $\lim_{t\to\infty} x(t) = 0$, an inherent property of a MRAC system. □

This new higher-order tracking error convergence property: $\lim_{t\to\infty} \frac{d^i e(t)}{dt^i} = 0$, for $i = 1, \ldots, n^*$, of MRAC, is extendable to adaptive nonlinear tracking control systems in which the controlled nonlinear plant has a defined relative degree $n^*$.

# Convergence of $\tilde{\theta}^T(t)\omega(t)$

For a MRAC system, we have defined

$$\theta^* = (\theta_1^{*T}, \theta_2^{*T}, \theta_{20}^*, \theta_3^*)^T \in R^{2\bar{n}} \tag{114}$$

$$\theta(t) = (\theta_1^T(t), \theta_2^T(t), \theta_{20}(t), \theta_3(t))^T \in R^{2\bar{n}} \tag{115}$$

$$\omega(t) = (\omega_1^T(t), \omega_2^T(t), y(t), r(t))^T \in R^{2\bar{n}} \tag{116}$$

$$\tilde{\theta}(t) = \theta(t) - \theta^*,\ e(t) = y(t) - y_m(t), \tag{117}$$

and derived the tracking error equation

$$\begin{aligned} e(t) &= \frac{k_p}{P_m(s)}[\tilde{\theta}^T\omega](t) \\ &= -k_p(\theta^{*T}\frac{1}{P_m(s)}[\omega](t) - \frac{1}{P_m(s)}[\theta^T\omega](t)), \end{aligned} \tag{118}$$

where $P_m(s)$ is a monic and stable polynomial of degree $n^*$.

We also defined the estimation error

$$\varepsilon(t) = e(t) + \rho(t)\xi(t) \tag{119}$$

where $\rho(t)$ is the estimate of $\rho^* = k_p$, and

$$\xi(t) = \theta^T(t)\zeta(t) - \frac{1}{P_m(s)}[\theta^T\omega](t) \tag{120}$$

$$\zeta(t) = \frac{1}{P_m(s)}[\omega](t). \tag{121}$$

This estimation error $\varepsilon(t)$, which is in its implementable (calculable) form, is inspired by the second equality of the tracking error equation (118):

$$\varepsilon(t) = e(t) - \left(-k_p(\theta^{*T}\frac{1}{P_m(s)}[\omega](t) - \frac{1}{P_m(s)}[\theta^T\omega](t))\right)|_{\theta^*=\theta,\rho^*=\rho}, \tag{122}$$

that is, $\varepsilon(t) = e(t) -$ the estimate of $-k_p(\theta^{*T}\frac{1}{P_m(s)}[\omega](t) - \frac{1}{P_m(s)}[\theta^T\omega](t))$.

The estimation error $\varepsilon(t)$ can be expressed as its theoretical form

$$\varepsilon(t) = k_p\tilde{\theta}^T(t)\zeta(t) + \tilde{\rho}(t)\xi(t),\ \tilde{\rho}(t) = \rho(t) - \rho^*. \tag{123}$$

In the MRAC literature, it was shown that $\lim_{t\to\infty}\varepsilon(t) = 0$, and $\lim_{t\to\infty} e(t) = 0$, that is, $\lim_{t\to\infty} k_p\tilde{\theta}^T(t)\zeta(t) + \tilde{\rho}(t)\xi(t) = 0$, and $\lim_{t\to\infty}\frac{k_p}{P_m(s)}[\tilde{\theta}^T\omega](t) = 0$.

How about $\lim_{t\to\infty} k_p\tilde{\theta}^T(t)\omega(t) = 0$?

Recently, some higher-order tracking properties of model reference adaptive control systems have been established in

G. Tao and G. Song, "Higher-order tracking properties of model reference adaptive control systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 11, pp. 3912–3918, 2018.

It is shown in this paper that a MRAC system ensures that the tracking error $e(t)$ satisfies: $\lim_{t\to\infty} \frac{d^i e(t)}{dt^i} = 0$, for $i = 0, 1, \ldots, n^* - 1$. If, in addition, $\dot{r}(t)$ is bounded, then $\lim_{t\to\infty} \frac{d^{n^*} e(t)}{dt^{n^*}} = 0$ also holds.

Hence, for a MRAC system with $e(t) = \frac{k_p}{P_m(s)}[\tilde{\theta}^T \omega](t)$, where $P_m(s)$ is a monic and stable polynomial of degree $n^*$, if $\dot{r}(t)$ is bounded, then $\lim_{t\to\infty} \frac{d^i e(t)}{dt^i} = 0$, for $i = 0, 1, \ldots, n^*$, that is, $\lim_{t\to\infty} k_p \tilde{\theta}^T(t)\omega(t) = \lim_{t\to\infty} P_m(s)[e](t) = 0$.

# Higher-Order Convergence of $\epsilon(t) = \tilde{\theta}^T(t)\phi(t)$

From a linear model

$$y(t) = \theta^{*T}\phi(t), \tag{124}$$

where $\theta^* \in R^{n_\theta}$ is an unknown parameter vector and $\phi(t) \in R^{n_\theta}$ is a known vector signal, we can use the normalized gradient algorithm (adaptive update law) to generate an estimate $\theta(t)$ of $\theta^*$:

$$\dot{\theta}(t) = -\frac{\Gamma\phi(t)\epsilon(t)}{m^2(t)},\ \theta(0) = \theta_0 \tag{125}$$

where $\Gamma = \Gamma^T > 0$ is a gain matrix, $\epsilon(t)$ is the estimation error defined as

$$\epsilon(t) = \theta^T(t)\phi(t) - y(t), \tag{126}$$

$m(t)$ is the normalization signal defined as

$$m(t) = \sqrt{1 + \alpha\phi^T(t)\phi(t)}, \tag{127}$$

with $\alpha > 0$ being a design parameter, and $\theta_0$ is an initial estimate of $\theta^*$.

This adaptive algorithm guarantees: (i) $\theta(t)$, $\dot{\theta}(t)$, $\frac{\epsilon(t)}{m(t)} \in L^\infty$; and (ii) $\dot{\theta}(t)$, $\frac{\epsilon(t)}{m(t)} \in L^2$.

If, in addition, $\phi(t) \in L^\infty (\Rightarrow m(t) \in L^\infty)$ and $\dot{\phi}(t) \in L^\infty (\Rightarrow \dot{\epsilon}(t) \in L^\infty)$, then $\lim_{t\to\infty} \epsilon(t) = 0$.

From the expressions of $\epsilon(t)$:

$$\epsilon(t) = \theta^T(t)\phi(t) - y(t) = \tilde{\theta}^T(t)\phi(t),\ \tilde{\theta}(t) = \theta(t) - \theta^*, \tag{128}$$

we can show that if, in addition, some order derivatives of $y(t)$ and $\phi(t)$ are bounded, then certain order derivatives of $\epsilon(t)$ also converge to 0, using the following lemma.

**Lemma**: If $\lim_{t\to\infty} f(t) = f_0$ has a constant limit $f_0$ and $\ddot{f}(t)$ is bounded, then $\lim_{t\to\infty} \dot{f}(t) = 0$.

For example, with $\lim_{t\to\infty} \epsilon(t) = 0$, if $\ddot{\epsilon}(t)$ is bounded, then $\lim_{t\to\infty} \dot{\epsilon}(t) = 0$.

To specify the conditions for $\ddot{\epsilon}(t)$ to be bounded, from $\epsilon(t) = \theta^T(t)\phi(t) - y(t)$, we have

$$\dot{\epsilon}(t) = \dot{\theta}^T(t)\phi(t) + \theta^T(t)\dot{\phi}(t) - \dot{y}(t) \tag{129}$$

$$\ddot{\epsilon}(t) = \ddot{\theta}^T(t)\phi(t) + \dot{\theta}^T(t)\dot{\phi}(t) + \dot{\theta}^T(t)\dot{\phi}(t) + \theta^T(t)\ddot{\phi}(t) - \ddot{y}(t). \tag{130}$$

From (125), we have

$$\ddot{\theta}(t) = -\Gamma\left(\frac{\dot{\phi}(t)(1+\alpha\phi^T(t)\phi(t)) - 2\alpha\phi^T(t)\dot{\phi}(t)\phi(t)}{(1+\alpha\phi^T(t)\phi(t))^2}\varepsilon(t) + \frac{\phi(t)}{1+\alpha\phi^T(t)\phi(t)}\dot{\varepsilon}\right). \tag{131}$$

Hence, if $\phi(t), \dot{\phi}(t), \ddot{\phi}(t), y(t), \dot{y}(t), \ddot{y}(t)$ are bounded, then $\ddot{\varepsilon}(t)$ is bounded, leading to $\lim_{t\to\infty}\dot{\varepsilon}(t) = 0$.

# Higher-Order Convergence of Indirect MRAC

## Higher-Order Tracking of Indirect MRAC

For an indirect MRAC system, the tracking error $e(t) = y(t) - y_m(t)$ also satisfies

$$e(t) = \frac{k_p}{P_m(s)}[\tilde{\theta}^T \omega](t), \tag{132}$$

where $k_p$ is the plant high frequency gain, $P_m(s)$ is a monic and stable polynomial of degree $n^*$ (the plant relative degree), and

$$\tilde{\theta}(t) = \theta(t) - \theta^* \tag{133}$$

$$\theta^* = (\theta_1^{*T}, \theta_2^{*T}, \theta_{20}^*, \theta_3^*)^T \in R^{2n} \tag{134}$$

$$\theta(t) = (\theta_1^T(t), \theta_2^T(t), \theta_{20}(t), \theta_3(t))^T \in R^{2n} \tag{135}$$

$$\omega(t) = (\omega_1^T(t), \omega_2^T(t), y(t), r(t))^T \in R^{2n}. \tag{136}$$

Hence, it can also be shown that an indirect MRAC system ensures that the tracking error $e(t)$ satisfies: $\lim_{t\to\infty} \frac{d^i e(t)}{dt^i} = 0$, for $i = 0, 1, \ldots, n^* - 1$. If, in addition, $\dot{r}(t)$ is bounded, then $\lim_{t\to\infty} \frac{d^{n^*} e(t)}{dt^{n^*}} = 0$ also holds.

## Higher-Order Convergence of Signal Estimation

Consider a linear time-invariant plant described by the differential equation

$$P(s)[y](t) = k_p Z(s)[u](t), \tag{137}$$

where $y(t) \in R$ and $u(t) \in R$ are the measured plant input and output,

$$P(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0, \tag{138}$$

$$Z(s) = s^m + z_{m-1}s^{m-1} + \cdots + z_1 s + z_0 \tag{139}$$

are polynomials in $s$ with $s$ being the time differentiation operator $s[x](t) = \dot{x}(t)$, and $p_i$, $i = 0, 1, \ldots, n-1$, $k_p$, and $z_i$, $i = 0, 1, \ldots, m-1$, with $n > m$, are the unknown constant parameters.

Choosing a stable polynomial $\Lambda_e(s) = s^n + \lambda_{n-1}^e s^{n-1} + \cdots + \lambda_1^e s + \lambda_0^e$ and defining the parameter and regressor vectors

$$\theta_p^* = [k_p z_0, k_p z_1, \ldots, k_p z_{m-1}, k_p,$$

$$-p_0, -p_1, \ldots, -p_{n-2}, -p_{n-1}]^T \in R^{n+m+1}, \tag{140}$$

$$\phi(t) = \Big[\frac{1}{\Lambda(s)}[u](t), \frac{s}{\Lambda_e(s)}[u](t), \ldots, \frac{s^{m-1}}{\Lambda_e(s)}[u](t), \frac{s^m}{\Lambda_e(s)}[u](t),$$
$$\frac{1}{\Lambda_e(s)}[y](t), \frac{s}{\Lambda_e(s)}[y](t), \ldots, \frac{s^{n-2}}{\Lambda_e(s)}[y](t), \frac{s^{n-1}}{\Lambda_e(s)}[y](t)\Big]^T, \tag{141}$$

we express the plant (137) as

$$y(t) - \frac{\Lambda_{n-1}(s)}{\Lambda_e(s)}[y](t) = \theta_p^{*T}\phi(t), \tag{142}$$

where $\Lambda_{n-1}(s) = \lambda_{n-1}^e s^{n-1} + \cdots + \lambda_1^e s + \lambda_0^e$.

Denoting $\theta_p(t)$ as the estimate of $\theta_p^*$, we define the estimation error

$$\varepsilon(t) = \theta_p^T(t)\phi(t) - y(t) + \frac{\Lambda_{n-1}(s)}{\Lambda_e(s)}[y](t), \tag{143}$$

which can be expressed as

$$\varepsilon(t) = \tilde{\theta}_p^T(t)\phi(t),\ \tilde{\theta}_p(t) = \theta_p(t) - \theta_p^*. \tag{144}$$

From (142), we have

$$y(t) = \frac{\Lambda_{n-1}(s)}{\Lambda_e(s)}[y](t) + \theta_p^{*T}\phi(t), \tag{145}$$

and define the estimate of $y(t)$ as

$$\hat{y}(t) = \frac{\Lambda_{n-1}(s)}{\Lambda_e(s)}[y](t) + \theta_p^T(t)\phi(t). \tag{146}$$

It follows that

$$\tilde{y}(t) \overset{\triangle}{=} \hat{y}(t) - y(t) = \tilde{\theta_p}^T(t)\phi(t) = \varepsilon(t), \tag{147}$$

so that

$$\lim_{t\to\infty}(\hat{y}(t) - y(t)) = 0. \tag{148}$$

From (145), we have

$$\dot{y}(t) = \frac{s\Lambda_{n-1}(s)}{\Lambda_e(s)}[y](t) + \theta_p^{*T}\dot{\phi}(t), \tag{149}$$

and define the estimate of $\dot{y}(t)$ as

$$\hat{\dot{y}}(t) = \frac{s\Lambda_{n-1}(s)}{\Lambda_e(s)}[y](t) + \theta_p^T(t)\dot{\phi}(t). \tag{150}$$

It follows that

$$\tilde{y}(t) \stackrel{\triangle}{=} \dot{\hat{y}}(t) - \dot{y}(t) = \tilde{\theta}_p^T(t)\dot{\phi}(t). \tag{151}$$

From $\varepsilon(t) = \tilde{\theta}_p^T(t)\phi(t)$, we have

$$\dot{\varepsilon}(t) = \dot{\tilde{\theta}}_p^T(t)\phi(t) + \tilde{\theta}_p^T(t)\dot{\phi}(t) \tag{152}$$

$$\ddot{\varepsilon}(t) = \ddot{\tilde{\theta}}_p^T(t)\phi(t) + \dot{\tilde{\theta}}_p^T(t)\dot{\phi}(t) + \dot{\tilde{\theta}}_p^T(t)\dot{\phi}(t) + \tilde{\theta}_p^T(t)\ddot{\phi}(t). \tag{153}$$

If an adaptive law without parameter projection for $\theta_p(t)$:

$$\dot{\theta}_p(t) = -\frac{\Gamma\phi(t)\varepsilon(t)}{1+\alpha\phi^T(t)\phi(t)},\ \Gamma = \Gamma^T > 0,\ \alpha > 0 \tag{154}$$

can achieve the objectives of the asymptotic tracking and signal boundedness of the MRAC system, then it can be verified that $\ddot{\varepsilon}(t)$ is bounded and $\ddot{\tilde{\theta}}_p(t)$ is bounded, so that, with $\lim_{t\to\infty}\varepsilon(t) = 0$, we have $\lim_{t\to\infty}\dot{\varepsilon}(t) = 0$. Then, since $\lim_{t\to\infty}\dot{\tilde{\theta}}_p(t) = 0$, it follows that $\lim_{t\to\infty}\tilde{\theta}_p^T(t)\dot{\phi}(t) = 0$, that is, $\lim_{t\to\infty}(\dot{\hat{y}}(t) - \dot{y}(t)) = 0$.

However, if the parameter projection applied to the adaptive law (154) (for the estimate of $k_p$, the $(m+1)$th element of $\theta_p(t)$) is in action, then the $(m+1)$th element of $\dot{\theta}_p(t)$ may be discontinuous at the parameter boundary of $k_p$ (its lower bound), so that the $(m+1)$th element of $\ddot{\theta}_p(t)$ may not be bounded. In this case, $\ddot{\varepsilon}(t)$ may not be bounded, and $\lim_{t\to\infty}\dot{\varepsilon}(t) = 0$ may not hold. Hence, from $\dot{\varepsilon}(t) = \dot{\tilde{\theta}}_p^T(t)\phi(t) + \tilde{\theta}_p^T(t)\dot{\phi}(t)$, we may not conclude $\lim_{t\to\infty}\tilde{\theta}_p^T(t)\dot{\phi}(t) = 0$ for $\dot{\hat{y}}(t) - \dot{y}(t) = \tilde{\theta}_p^T(t)\dot{\phi}(t)$, even if $\lim_{t\to\infty}\dot{\tilde{\theta}}_p(t) = 0$ (as $\lim_{t\to\infty}\varepsilon(t) = 0$).

# Analysis of "Bursting Phenomenon" in Robust Adaptive Control

As illustrated in [134] that adaptive control systems with a fixed $\sigma$-modification may show some bursting phenomenon. Consider the adaptive system analyzed on page 232 (for Theorem 5.8) with $\sigma_1(t) = \sigma_0$ being a constant in (5.198):

$$\dot{\theta}(t) = -\mathrm{sign}[k_p]\Gamma\omega(t)e(t) - \Gamma\sigma_1(t)\theta(t),\ t \geq 0$$

(that is, with a fixed $\sigma$-modification). For $V$ given in (5.43): $V = e^2 + |k_p|\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}$, we have from (5.204) that

$$\begin{aligned}
\dot{V} &\leq -a_m e^2(t) + \frac{\bar{d}^2(t)}{a_m} - 2|k_p|\sigma_0\tilde{\theta}^T(t)\theta(t) \\
&= -a_m e^2(t) + \frac{\bar{d}^2(t)}{a_m} - 2|k_p|\sigma_0\tilde{\theta}^T(t)\tilde{\theta}(t) + 2|k_p|\sigma_0\tilde{\theta}^T(t)\theta^* \\
&\leq -a_m e^2(t) + \frac{\bar{d}^2(t)}{a_m} - |k_p|\sigma_0\tilde{\theta}^T(t)\tilde{\theta}(t) + |k_p|\sigma_0\theta^{*T}\theta^*. \qquad (155)
\end{aligned}$$

For $\Gamma = \gamma I$ and $a_m > \gamma\sigma_0$ (that is, with a small $\sigma_0$), we have

$$\dot{V} \leq -\gamma\sigma_0 V + \frac{\bar{d}^2(t)}{a_m} + |k_p|\sigma_0\theta^{*T}\theta^* \qquad (156)$$

which, for $|\bar{d}(t)| \leq d_0$, leads to

$$\lim_{t\to\infty} V(t) \leq \frac{d_0^2}{a_m\gamma\sigma_0} + \frac{|k_p|}{\gamma}\theta^{*T}\theta^*. \qquad (157)$$

This implies that the upper bound for the tracking error $e(t) = y(t) - y_m(t)$, for $d(t) = d_0 = 0$ (in the absence of disturbances $d(t)$), may be as large as $\frac{|k_p|}{\gamma}\theta^{*T}\theta^*$, independent of $\sigma_0$. We already knew that for $d(t) = 0$ and $\sigma_1(t) = \sigma_0 = 0$, the adaptive control system ensures that $\lim_{t\to\infty} e(t) = 0$. Now from the above analysis we know that for a small $\sigma_0 \neq 0$, the error bound on $|e(t)|$ can be up to $\frac{|k_p|}{\gamma}\theta^{*T}\theta^*$. On the other hand, from

$$\dot{V} \leq -a_m e^2(t) + \frac{\bar{d}^2(t)}{a_m} - 2|k_p|\sigma_0\tilde{\theta}^T(t)\theta(t) \qquad (158)$$

and the adaptive control system signal boundedness, similar to (5.204), we have

$$\int_{t_1}^{t_2} e^2(t)\,dt \leq \gamma_0 + k_0(t_2 - t_1)\bar{d}_0^2 + c_0(t_2 - t_1)\sigma_0 \qquad (159)$$

for some constant $c_0 > 0$, where the $\sigma_0$ related term is due to using a fixed $\sigma$-modification $\sigma_1(t) = \sigma_0$, instead of the switching $\sigma$-modification which leads to (5.205) and in turn to (5.204). The inequality (159) implies that, when $d_0 = 0$ (in the absence of disturbances), we have the mean error

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} e^2(t)\, dt \le \frac{\gamma_0}{t_2 - t_1} + c_0 \sigma_0 \tag{160}$$

which is of the magnitude of $c_0\sigma_0$, but the absolute error $|e(t)|$, as from (157), could be as large as $\frac{|k_p|}{\gamma}\theta^{*T}\theta^*$, independent of $\sigma_0$. This is the so-called "bursting phenomenon" of robust adaptive control with a fixed $\sigma$-modification: the tracking error $e(t)$ may go to a large value independent of $\sigma_0$ for a small interval of time but in the mean sense the error $e(t)$ is of the order of $\sigma_0$. This analytically explains what was observed in the simulation results [134].

# Robust MRAC with A Switching σ-Modification

To derive (5.224), for $V(\tilde{\theta},\tilde{\rho}) = |\rho^*|\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \gamma^{-1}\tilde{\rho}^2$, using

$$\epsilon(t) = \rho^*\tilde{\theta}^T(t)\zeta(t) + \tilde{\rho}(t)\xi(t) + \mu\eta(t),\ \eta(t) = \Delta(s)[u](t) \tag{161}$$

we have

$$\begin{aligned}
\dot{V} &= -2\frac{\epsilon^2(t)}{m^2(t)} + 2\frac{\epsilon(t)\eta(t)}{m^2(t)} - 2\sigma_1(t)|\rho^*|\tilde{\theta}^T(t)\theta(t) - 2\sigma_2(t)\tilde{\rho}(t)\rho(t)\\
&= -\frac{\epsilon^2(t)}{m^2(t)} - \left(\frac{\epsilon(t)}{m(t)} - \mu\frac{\eta(t)}{m(t)}\right)^2 + \mu^2\frac{\eta^2(t)}{m^2(t)}\\
&\quad -2\sigma_1(t)|\rho^*|\tilde{\theta}^T(t)\theta(t) - 2\sigma_2(t)\tilde{\rho}(t)\rho(t)\\
&\leq -\frac{\epsilon^2(t)}{m^2(t)} + \mu^2\frac{\eta^2(t)}{m^2(t)} - 2\sigma_1(t)|\rho^*|\tilde{\theta}^T(t)\theta(t) - 2\sigma_2(t)\tilde{\rho}(t)\rho(t)
\end{aligned} \tag{162}$$

which is (5.224). Since $\frac{|\eta(t)|}{m(t)} \leq b_0$ for some constant $b_0 > 0$, we have $\dot{V} < 0$ if

$$2\sigma_1(t)|\rho^*|\tilde{\theta}^T(t)\theta(t) + 2\sigma_2(t)\tilde{\rho}(t)\rho(t) > \mu^2 b_0^2. \tag{163}$$

For $\tilde{\theta} = \theta - \theta^*$ and $\tilde{\rho} = \rho - \rho^*$, by definition, we have

$$\sigma_1(t)|\rho^*|\tilde{\theta}^T(t)\theta(t) \geq 0,\ \sigma_2(t)\tilde{\rho}(t)\rho(t) \geq 0,\ t \geq 0 \tag{164}$$

$$\lim_{\|\theta\|_2\to\infty} \tilde{\theta}^T\theta = \infty,\ \lim_{|\rho|\to\infty} \tilde{\rho}\rho = \infty. \tag{165}$$

This implies that there exist constants $\theta^0 > 0$ and $\rho^0 > 0$ such that $\|\theta(t)\|_2 \geq \theta^0$ or/and $|\rho(t)| \geq \rho^0$ implies that $\dot{V} < 0$. Hence, the boundedness of $\theta(t)$ and $\rho(t)$ is ensured. One choice of such ($\theta^0$, $\rho^0$) is

$$\begin{aligned}
\theta^0 &= \max\{2M_1, \sqrt{\frac{1}{4\sigma_{10}}(2\mu^2 b_0^2 + \|\theta^*\|_2^2 + \rho^{*2})} + \frac{1}{2}\|\theta^*\|\}\\
\rho^0 &= \max\{2M_2, \sqrt{\frac{1}{4\sigma_{20}}(2\mu^2 b_0^2 + \|\theta^*\|_2^2 + \rho^{*2})} + \frac{1}{2}|\rho^*|\}.
\end{aligned} \tag{166}$$

# A Discrete-time MRAC System Example

Consider the first-order plant

$$y(t+1) = a_p y(t) + b_p u(t) \tag{167}$$

with two unknown parameters $a_p$ and $b_p$ and choose the reference model

$$y_m(t+1) = -a_m y_m(t) + b_m r(t) \tag{168}$$

with $|a_m| < 1$ for stability. We use the adaptive controller structure

$$u(t) = k_1(t) y(t) + k_2(t) r(t), \tag{169}$$

where $k_1(t)$ and $k_2(t)$ are estimates of the unknown parameters

$$k_1^* = \frac{-a_p - a_m}{b_p},\ k_2^* = \frac{b_m}{b_p}. \tag{170}$$

In this case the tracking error equation becomes

$$e(t+1) = -a_m e(t) + b_p \tilde{k}_1(t) y(t) + b_p \tilde{k}_2(t) r(t), \tag{171}$$

where $\tilde{k}_1(t) = k_1(t) - k_1^*$ and $\tilde{k}_2(t) = k_2(t) - k_2^*$.

Defining $\rho^* = b_p$ and

$$\theta^* = [k_1^*, k_2^*]^T,\ \theta = [k_1, k_2]^T, \tag{172}$$

$$\omega(t) = [y(t), r(t)]^T \tag{173}$$

and introducing the filtered vector signal

$$\zeta(t) = \frac{1}{z+a_m}[\omega](t) = \left[\frac{1}{z+a_m}[y](t), \frac{1}{z+a_m}[r](t)\right]^T = [\zeta_1(t), \zeta_2(t)]^T, \tag{174}$$

where, as a notation, $\zeta_1(t) = \frac{1}{z+a_m}[y](t)$ denotes the output of the system with transfer function $\frac{1}{z+a_m}$ and input $y(t)$ (it satisfies the equation: $\zeta_1(t+1) = -a_m \zeta_1(t) + y(t)$, for generating $\zeta_1(t)$ from $y(t)$), we rewrite (171) as

$$\begin{aligned} e(t) &= \frac{\rho^*}{z+a_m}[\theta^T \omega - \theta^{*T} \omega](t) \\ &= \rho^* \left( \frac{1}{z+a_m}[\theta^T \omega](t) - \theta^{*T} \zeta(t) \right). \end{aligned} \tag{175}$$

The design task is to find adaptive laws to update the parameter estimates $\theta(t)$ and $\rho(t)$ (which is an estimate of $\rho^*$) such that the estimation error

$$\epsilon(t) = e(t) - \rho(t)\left(\frac{1}{z+a_m}[\theta^T\omega](t) - \theta^T(t)\zeta(t)\right) \tag{176}$$

is small in some sense. With (175), this error can be expressed as

$$\epsilon(t) = \rho^*\tilde{\theta}^T(t)\zeta(t) + \tilde{\rho}(t)\xi(t), \tag{177}$$

where

$$\xi(t) = \theta^T(t)\zeta(t) - \frac{1}{z+a_m}[\theta^T\omega](t) \tag{178}$$

$$\tilde{\theta}(t) = \theta(t) - \theta^*,\ \tilde{\rho}(t) = \rho(t) - \rho^*. \tag{179}$$

We choose the gradient adaptive laws for $\theta(t)$ and $\rho(t)$:

$$\theta(t+1) = \theta(t) - \frac{\text{sign}[b_p]\Gamma\epsilon(t)\zeta(t)}{m^2(t)},\ 0 < \Gamma = \Gamma^T < \frac{2}{b_p^0}I_2, \tag{180}$$

$$\rho(t+1) = \rho(t) - \frac{\gamma\epsilon(t)\xi(t)}{m^2(t)},\ 0 < \gamma < 2, \tag{181}$$

where $\text{sign}[b_p]$ is the sign of $b_p$, $b_p^0 \geq |b_p|$ is a known upper bound on $|b_p|$, and

$$m(t) = \sqrt{1 + \zeta^T(t)\zeta(t) + \xi^2(t)}. \tag{182}$$

The stability analysis of this MRAC system is given in Section 6.3.1.

# Indirect Observer Based Adaptive Control

Consider a continuous-time linear multivariable time-invariant plant

$$y(t) = G(s)[u](t),\ u(t) \in R^M,\ y(t) \in R^M,\ M \geq 1, \tag{183}$$

where $G(s)$ is an $M \times M$ strictly proper rational transfer matrix which can be expressed as

$$G(s) = D^{-1}(s)N(s) = C_g(sI - A_g)^{-1}B_g, \tag{184}$$

where, for the observability index $\nu$ of $G(s)$ (its minimal realization),

$$D(s) = s^{\nu} I_M + A_{\nu-1}s^{\nu-1} + \cdots + A_1 s + A_0, \tag{185}$$

$$N(s) = B_m s^m + \cdots + B_1 s + B_0, \tag{186}$$

$$A_g = \begin{bmatrix} -A_{\nu-1} & I_M & 0 & \cdots & \cdots & 0 \\ -A_{\nu-2} & 0 & I_M & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -A_1 & 0 & \cdots & \cdots & 0 & I_M \\ -A_0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \in R^{M\nu \times M\nu}, \tag{187}$$

$$B_g = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_p \end{bmatrix} \in R^{M\nu \times M},\ B_p = \begin{bmatrix} B_m \\ \vdots \\ B_1 \\ B_0 \end{bmatrix} \in R^{M(m+1)\times M}, \tag{188}$$

$$C_g = \begin{bmatrix} I_M & 0 & \cdots & 0 & 0 \end{bmatrix} \in R^{M \times M\nu} \tag{189}$$

for the $M \times M$ identity matrix $I_M$ and some $M \times M$ parameter matrices $A_i$, $i = 0, 1, \ldots, \nu - 1$, and $B_j$, $j = 0, 1, \ldots, m$, $m \leq \nu - 1$.

In view of (184), the plant (183) can be expressed as

$$\dot{x}(t) = Ax(t) + A_p y(t) + \begin{bmatrix} 0 \\ B_p \end{bmatrix} u(t),\ y(t) = C_g x(t), \tag{190}$$

where $A \in R^{M\nu \times M\nu}$ and $A_p \in R^{M\nu \times M}$ are

$$A = \begin{bmatrix} 0 & I_M & 0 & \cdots & \cdots & 0 \\ 0 & 0 & I_M & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & I_M \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix},\ A_p = \begin{bmatrix} -A_{\nu-1} \\ -A_{\nu-2} \\ \vdots \\ -A_1 \\ -A_0 \end{bmatrix}. \tag{191}$$

**Control objective**. The control objective is to design an adaptive feedback control signal $u(t)$ to make the plant output $y(t)$ tracks the output $y_m(t)$ of the reference model system

$$y_m(t) = W_m(s)[r](t),\ W_m(s) = \xi_m^{-1}(s), \tag{192}$$

where $\xi_m(s)$ is the modified left interactor matrix of $G(s)$: $\lim_{s\to\infty} \xi_m(s)G(s) = K_p$ is finite and nonsingular.

**Adaptive parameter estimation**. Based on the input-output model of the plant: $D(s)[y](t) = N(s)[u](t)$, we can design an adaptive parameter estimation scheme to generate adaptive estimates $\hat{A}_i$ and $\hat{B}_j$ of the coefficient matrices $A_i$ and $B_j$ of $D(s)$ and $N(s)$. From the parameter estimates $\hat{A}_i$ and $\hat{B}_j$, we can obtain the adaptive estimates $\hat{A}_p$ and $\hat{B}_p$ of the parameter matrices $A_p$ and $B_p$, and the adaptive estimates $\hat{A}_g$ and $\hat{B}_g$ of the parameter matrices $A_g$ and $B_g$.

**Adaptive state observer**. With adaptive estimates $\hat{A}_p$ and $\hat{B}_p$, we construct an adaptive state observer for the plant (190) as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \hat{A}_p y(t) + \begin{bmatrix} 0 \\ \hat{B}_p \end{bmatrix} u(t) + L(y(t) - \hat{y}(t)),\ \hat{y}(t) = C_g\hat{x}(t), \tag{193}$$

where $L \in R^{M\nu\times M}$ is such that $A - LC_g$ is a desired stable matrix. For the state estimation error $\tilde{x}(t) = \hat{x}(t) - x(t)$, it follows that

$$\dot{\tilde{x}}(t) = (A - LC_g)\tilde{x}(t) + \tilde{A}_p(t)y(t) + \begin{bmatrix} 0 \\ \tilde{B}_p(t) \end{bmatrix} u(t),\ \tilde{y}(t) = C_g\tilde{x}(t) \tag{194}$$

where $\tilde{y}(t) = \hat{y}(t) - y(t)$, $\tilde{A}_p(t) = \hat{A}_p(t) - A_p$ and $\tilde{B}_p(t) = \hat{B}_p(t) - B_p$.

**Adaptive control law**. The adaptive control law is an indirect-design (that is, its parameters are calculated from plant parameter estimates) and explicit-observer (that is, it uses the state estimate $\hat{x}(t)$ for feedback control) based control law:

$$u(t) = K_1^T(t)\hat{x}(t) + K_2(t)r(t), \tag{195}$$

where $K_1(t) \in R^{M\nu\times M}$ and $K_2(t) \in R^{M\times}$ are parameter matrices which satisfy

$$C_g(\lambda I - \hat{A}_g - \hat{B}_g K_1^T)^{-1}\hat{B}_g K_2 = W_m(\lambda), \tag{196}$$

point-wise in the time variable $t$, where $\hat{A}_g(t)$ and $\hat{B}_g(t)$ are the on-line estimates of $A_g$ and $B_g$. It can be shown that such solutions $K_1$ and $K_2$ exist if the estimated plant $(\hat{A}_g, \hat{B}_g, C_g)$ has $\xi_m(s)$ as its modified left interactor matrix point-wise, that is, $\lim_{\lambda\to\infty} \xi_m(\lambda)C_g(\lambda I - \hat{A}_g)^{-1}\hat{B}_g = \hat{K}_p$ is finite and non-singular, for each time $t$ as $\hat{A}_g = \hat{A}_p(t)$ and $\hat{B}_g = \hat{B}_p(t)$ (note that $\hat{K}_p$ may be a function of time $t$).

In the single-input single-output case with $M = 1$, such a condition becomes: $\hat{B}_m(t) = \hat{b}_m(t) \neq 0$ for any $t$, which can be easily ensured by using a parameter projection based adaptive parameter estimator. In this case, $\xi_m(\lambda) = \lambda^{\nu - m}$ and $\hat{K}_p = \hat{b}_m(t)$, for $\nu = n$ being the plant order.

# Adaptive Control of $\dot{x}(t) = Ax(t) + B\Lambda u(t) + B\Theta^*\phi(x)$

Consider an $M$-input linear time-invariant plant

$$\dot{x}(t) = Ax(t) + B\Lambda u(t) + B\Theta^*\phi(x),\ x(t) \in R^n,\ u(t) \in R^M, \tag{197}$$

where ($A$, $B$) is a known and controllable pair, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_M\}$ is an unknown and diagonal matrix with $\lambda_i \neq 0$ for $i = 1, 2, \ldots, M$, $\Theta^*$ is an unknown matrix, and $\phi(x)$ is a known vector.

With ($A$, $B$) known and controllable, we can find known constant matrices $K_{10} \in R^{n\times M}$ and $K_{20} \in R^{M\times M}$ such that

$$A_m = A + BK_{10}^T,\ B_m = BK_{20} \tag{198}$$

are known with $A_m$ being stable, for constructing a good reference model system

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t),\ x_m(t) \in R^n,\ r(t) \in R^M, \tag{199}$$

where $r(t)$ is a bounded and piecewise continuous reference input.

We first introduce the parameter matrices

$$K_1^{*T} = \Lambda^{-1}K_{10}^T,\ K_2^* = \Lambda^{-1}K_{20},\ K_3^* = -\Lambda^{-1}\Theta^*. \tag{200}$$

Then, for $K_1$, $K_2$ and $K_3$ being the estimates of $K_1^*$, $K_2^*$ and $K_3^*$, we use the adaptive controller

$$u(t) = K_1^T(t)x(t) + K_2(t)r(t) + K_3(t)\phi(x). \tag{201}$$

From the definitions of $K_1^*$, $K_2^*$ and $K_3^*$, we have

$$B\Lambda(K_1^{*T}x(t) + K_2^*r(t) + K_3^*\phi(x)) = B(K_{10}^T x(t) + K_{20}r(t) - \Theta^*\phi(x)). \tag{202}$$

Introducing the parameter errors

$$\tilde{K}_1 = K_1 - K_1^*,\ \tilde{K}_2 = K_2 - K_2^*,\ \tilde{K}_3 = K_3 - K_3^* \tag{203}$$

we express the control signal from (201) as

$$u(t) = \tilde{K}_1^T(t)x(t) + \tilde{K}_2(t)r(t) + \tilde{K}_3(t)\phi(x) + K_1^{*T}x(t) + K_2^*r(t) + K_3^*\phi(x) \tag{204}$$

and, in view of (198) and (202), the closed-loop system as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\Lambda(\tilde{K}_1^T(t)x(t) + \tilde{K}_2(t)r(t) + \tilde{K}_3(t)\phi(x)) \\ &\quad + B\Lambda(K_1^{*T}x(t) + K_2^*r(t) + K_3^*\phi(x)) + B\Theta^*\phi(x) \\ &= A_m x(t) + B_m r(t) + B\Lambda(\tilde{K}_1^T(t)x(t) + \tilde{K}_2(t)r(t) + \tilde{K}_3(t)\phi(x)).\end{aligned} \tag{205}$$

Then, in view of this equation and (199), the tracking error $e(t) = x(t) - x_m(t)$ satisfies

$$\dot{e}(t) = A_m e(t) + B\Lambda(\tilde{K}_1^T(t)x(t) + \tilde{K}_2(t)r(t) + \tilde{K}_3(t)\phi(x)). \tag{206}$$

Introducing $\theta_i^*$ such that $\theta_i^{*T}$ is the $i$th row of $[K_1^{*T}, K_2^*, K_3^*]$, $i = 1,2,\ldots,M$, letting $\theta_i$ be the estimate of $\theta_i^*$ and $\tilde{\theta}_i(t) = \theta_i(t) - \theta_i^*$, $i = 1,2,\ldots,M$, and defining

$$\omega(t) = [x^T(t), r^T(t), \phi^T(x)]^T, \tag{207}$$

we express (206) as

$$\dot{e}(t) = A_m e(t) + B\Lambda \begin{bmatrix} \tilde{\theta}_1^T(t)\omega(t) \\ \tilde{\theta}_2^T(t)\omega(t) \\ \vdots \\ \tilde{\theta}_{M-1}^T(t)\omega(t) \\ \tilde{\theta}_M^T(t)\omega(t) \end{bmatrix}. \tag{208}$$

Letting $P = P^T > 0$ satisfy

$$PA_m + A_m^T P = -Q \tag{209}$$

for a chosen constant matrix $Q \in R^{n\times n}$ such that $Q = Q^T > 0$, and $\bar{e}_i(t)$ be the $i$th component of $e^T(t)PB$, $i = 1,2,\ldots,M$, we design the adaptive law for $\theta_i(t)$ as

$$\dot{\theta}_i(t) = -\text{sign}[\lambda_i]\Gamma_i \bar{e}_i(t)\omega(t), \tag{210}$$

where $\Gamma_i = \Gamma_i^T > 0$ is a chosen constant adaptation gain matrix, and sign$[\lambda_i]$ is the sign of $\lambda_i$, $i = 1,2,\ldots,M$.

To analyze the adaptive control system performance, we consider the positive definite function

$$V(e, \tilde{\theta}_i, i = 1,2,\ldots,M) = e^T Pe + \sum_{i=1}^{M} |\lambda_i| \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i \tag{211}$$

and derive its time derivative along the trajectory of (208) and (210) as

$$\begin{aligned} \dot{V} &= 2e^T(t)P\dot{e}(t) + 2\sum_{i=1}^{M} |\lambda_i| \tilde{\theta}_i^T(t)\Gamma_i^{-1}\dot{\theta}_i(t) \\ &= 2e^T(t)PA_m e(t) + 2\sum_{i=1}^{M} \bar{e}_i(t)\lambda_i \tilde{\theta}_i^T(t)\omega(t) + 2\sum_{i=1}^{M} |\lambda_i| \tilde{\theta}_i^T(t)\Gamma_i^{-1}\dot{\theta}_i(t) \\ &= -e^T(t)Qe(t). \end{aligned} \tag{212}$$

From this result, it follows that $x(t)$ and $\theta_i(t)$ are bounded and $e(t) \in L^2$, and in turn, that $u(t)$ is bounded, and so is $\dot{e}(t)$, so that $\lim_{t\to\infty} e(t) = 0$.

# Parameters of $u(t) = \Theta_1^{*T}\omega_1(t) + \Theta_2^{*T}\omega_2(t) + \Theta_{20}^* y(t) + \Theta_3^*\xi(z)y(t)$

For multivariable MRAC, the parameters $\Theta_1^*$, $\Theta_2^*$, $\Theta_{20}^*$ and $\Theta_3^*$ of the matching equation

$$u(t) = \Theta_1^{*T}\omega_1(t) + \Theta_2^{*T}\omega_2(t) + \Theta_{20}^* y(t) + \Theta_3^*\xi(z)y(t) \tag{213}$$

are the nominal controller parameters. For discrete-time MRAC where $\Lambda(z) = z^{n_c}$ for generating $\omega_1(t)$ and $\omega_2(t)$, such parameters can be calculated from the following procedure.

For a given system transfer matrix $G(z) = A^{-1}(z^{-1})B(z^{-1})$ and an interactor matrix $\xi(z)$ such that $\mathrm{rlim}_{z\to\infty}\xi(z)G(z) = K_p$ finite and nonsingular, where

$$A(z^{-1}) = I + A_1 z^{-1} + \cdots + A_{n_a} z^{-n_a},\ B(z^{-1}) = B_1 z^{-1} + \cdots + B_{n_b} z^{-n_b} \tag{214}$$

$$\xi(z) = \xi_0 z^d + \xi_1 z^{d-1} + \cdots + \xi_{d-1} z, \tag{215}$$

we can solve the equation

$$\xi(z) = F(z)A(z^{-1}) + H(z^{-1}) \tag{216}$$

for some matrix functions

$$F(z) = F_0 z^q + F_1 z^{d-1} + \cdots + F_{d-1} z \tag{217}$$

$$H(z^{-1}) = H_0 + H_1 z^{-1} + \cdots + H_{n_h} z^{-n_h} \tag{218}$$

for some $n_h > 0$ to be determined, that is,

$$\begin{aligned} &\xi_0 z^d + \xi_1 z^{d-1} + \cdots + \xi_{d-1} z \\ &= (F_0 z^d + F_1 z^{d-1} + \cdots + F_{d-1} z)(I + A_1 z^{-1} + \cdots + A_{n_a} z^{-n_a}) \\ &\quad + H_0 + H_1 z^{-1} + \cdots + H_{n_h} z^{-n_h}. \end{aligned} \tag{219}$$

The solutions $F_i$ and $H_j$ are from the iterative procedure

$$F_0 = \xi_0,\ F_1 + F_0 A_1 = \xi_1,\ \ldots \tag{220}$$

where $n_g$ depends on $n_a$ and $n_b$.

Then, we define

$$\alpha(z^{-1}) = H(z^{-1}) = \alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_{n_g} z^{-n_g} \tag{221}$$

with $\alpha_i = H_i$, and

$$\beta(z^{-1}) = F(z)B(z^{-1}). \tag{222}$$

From $\xi(z) = F(z)A(z^{-1}) + H(z^{-1})$, we have

$$\xi(z) - H(z^{-1}) = F(z)B(z^{-1})B^{-1}(z^{-1})A(z^{-1}) = \beta(z^{-1})G^{-1}(z) \tag{223}$$

that is,

$$G(z) = (\xi(z) - \alpha(z^{-1}))^{-1}\beta(z^{-1}). \tag{224}$$

Since $\lim_{z\to\infty} \xi(z)G(z) = K_p$ finite and nonsingular, we have

$$\lim_{z\to\infty} \beta(z^{-1}) = K_p \tag{225}$$

which implies that

$$\beta(z^{-1}) = \beta_0 + \beta_1 z^{-1} + \cdots + \beta_{n_\beta} z^{-n_\beta} \tag{226}$$

with $\beta_0 = K_p$ and $n_\beta$ determined from $\beta(z^{-1}) = F(z)B(z^{-1})$.

From $A(z^{-1})y(t) = B(z^{-1})u(t)$, we have $F(z)A(z^{-1})y(t) = F(z)B(z^{-1})u(t)$, and from $\xi(z) = F(z)A(z^{-1}) + H(z^{-1})$, we have

$$\begin{aligned} \xi(z)y(t) &= F(z)A(z^{-1})y(t) + H(z^{-1})y(t) = F(z)B(z^{-1})u(t) + H(z^{-1})y(t) \\ &= \alpha(z^{-1})y(t) + \beta(z^{-1})u(t). \end{aligned} \tag{227}$$

Comparing this equation with

$$\xi(z)y(t) = \Theta_3^{*-1}u(t) - \Theta_3^{*-1}\Theta_1^{*T}\omega_1(t) - \Theta_3^{*-1}\Theta_2^{*T}\omega_2(t) - \Theta_3^{*-1}\Theta_{20}^{*}y(t) \tag{228}$$

with $\Lambda(z) = z^{n_c}$, we can obtain $\Theta_1^*$, $\Theta_2^*$, $\Theta_{20}^*$ and $\Theta_3^* = K_p^{-1}$, from $\alpha(z^{-1})$ and $\beta(z^{-1})$.

**Example 1** Consider the system transfer matrix

$$G_0(z) = \begin{bmatrix} \frac{z^{-3}}{1+z^{-1}} & \frac{3z^{-3}}{1+4z^{-1}} \\ \frac{z^{-2}}{1+z^{-1}} & \frac{2z^{-2}}{1+3z^{-1}} \end{bmatrix}. \tag{229}$$

Its interactor matrix and high frequency gain matrix are

$$\xi(z) = \begin{bmatrix} z^3 & 0 \\ 0 & z^2 \end{bmatrix} = \xi_0 z^3 + \xi_1 z^2,\ K_p = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}. \tag{230}$$

It can be found that

$$A(z^{-1}) = \begin{bmatrix} (1+z^{-1})(1+4z^{-1}) & 0 \\ 0 & (1+z^{-1})(1+3z^{-1}) \end{bmatrix} \tag{231}$$

$$B(z^{-1}) = \begin{bmatrix} z^{-3}(1+4z^{-1}) & 3z^{-3}(1+z^{-1}) \\ z^{-2}(1+3z^{-1}) & 2z^{-2}(1+z^{-1}) \end{bmatrix}. \tag{232}$$

Hence, $n_a = 2$, $n_b = 4$, $d = 3$, and $F(z) = F_0 z^3 + F_1 z^2 + F_2 z$ in (217) with $F_i \in R^{2\times 2}$. For this example, $B_1 = 0$ and $\xi_2 = 0$ are 2 by 2 zero matrices. Since the lowest order in $(F_0 z^3 + F_1 z^2 + F_2 z)(I + A_1 z^{-1} + A_2 z^{-2})$ is $z^{-1}$, we have $n_h = 1$, so that (219) becomes

$$\xi_0 z^3 + \xi_1 z^2 = (F_0 z^3 + F_1 z^2 + F_2 z)(I + A_1 z^{-1} + A_2 z^{-2}) + H_0 + H_1 z^{-1}. \tag{233}$$

From this equation, we can find: $F_0, F_1, F_2, H_0, H_1$. Then, we define

$$\alpha(z^{-1}) = H(z^{-1}) = \alpha_0 + \alpha_1 z^{-1} \tag{234}$$

$$\beta(z^{-1}) = F(z)B(z^{-1}) = (F_0 z^3 + F_1 z^2 + F_2 z)(B_1 z^{-1} + B_2 z^{-2} + B_3 z^{-3} + B_4 z^{-4}). \tag{235}$$

It should turn out that

$$F_0 B_1 = F_0 B_2 = F_1 B_1 = 0 \tag{236}$$

$$F_0 B_3 + F_1 B_2 + F_2 B_1 = K_p = \beta_0 \tag{237}$$

Since $B_1 = 0$, we have $F_0 B_2 = 0$ and $F_0 B_3 + F_1 B_2 = K_p$. It can be seen from (235) that

$$\beta(z^{-1}) = \beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2} + \beta_3 z^{-3}, \tag{238}$$

that is, $n_\beta = 3$. The term $\beta(z^{-1})u(t)$ has the form

$$\beta(z^{-1})u(t) = \beta_0 + \beta_1 u(t-1) + \beta_2 u(t-2) + \beta_3 u(t-3). \tag{239}$$

For $G(z)$ in this example, it can be found the system order is $n = 5$ and its observability index $\nu \le n-1 = 4$, so that the order of $\Lambda(z)$ in (227) is $n_c = \nu - 1 = 3$, which means $\omega_1(t)$ depends on $u(t-1)$, $u(t-2)$ and $u(t-3)$, as contained in the term $\beta(z^{-1})u(t)$.

Finally, with $\alpha(z^{-1})$ and $\beta(z^{-1})$ specified, from the matching equations (227) and (228):

$$\xi(z)y(t) = \alpha(z^{-1})y(t) + \beta(z^{-1})u(t) \tag{240}$$

$$\xi(z)y(t) = \Theta_3^{*-1}u(t) - \Theta_3^{*-1}\Theta_1^{*T}\omega_1(t) - \Theta_3^{*-1}\Theta_2^{*T}\omega_2(t) - \Theta_3^{*-1}\Theta_{20}^{*}y(t) \tag{241}$$

we can find the parameters $\Theta_1^* \in R^{6\times 2}$, $\Theta_2^* \in R^{6\times 2}$, $\Theta_{20}^* \in R^{2\times 2}$ and $\Theta_3^* = \beta_0^{-1}$:

$$\Theta_1^* = [\beta_3, \beta_2, \beta_1]^T,\ \Theta_2^* = [\alpha_3, \alpha_2, \alpha_1]^T,\ \Theta_{20} = \alpha_0. \tag{242}$$

Since the term $\alpha(z^{-1})y(t)$ does not contain $y(-2)$ and $y(-3)$, $\alpha_3 = \alpha_2 = 0$ (the $2\times 2$ zero matrix), the first $4\times 2$ part of $\Theta_2^* \in R^{6\times 2}$ is the $4\times 2$ zero matrix. □

# Parametrization of (9.102)

To define the parameters $\Theta_i^*$, $i = 1,2,20,3$, to satisfy (9.105):

$$\begin{aligned} &\Theta_1^{*T} A(D)P_0(D) + (\Theta_2^{*T} A(D) + \Theta_{20}^* \Lambda(D))Z_0(D) \\ &= \Lambda(D)(P_0(D) - \Theta_3^* \xi_m(D)Z_0(D)), \end{aligned} \tag{243}$$

for the nominal version of the multivariable MRAC controller (9.102), we consider an equivalent version (see (244) below) of (9.105) by dividing it from the right by $P_0^{-1}(D)$ and use the left matrix-fraction description of $G_0(D) = P_l^{-1}(D)Z_l(D)$ such that $G_0(D) = P_l^{-1}(D)Z_l(D) = Z_0(D)P_0^{-1}(D)$ (as used in the proof of Lemma 9.3), to obtain

$$\begin{aligned} &\Theta_1^{*T} A(D) + (\Theta_2^{*T} A(D) + \Theta_{20}^* \Lambda(D))P_l^{-1}(D)Z_l(D) \\ &= \Lambda(D)(I - \Theta_3^* \xi_m(D)P_l^{-1}(D)Z_l(D)). \end{aligned} \tag{244}$$

Expressing $\Lambda(D)\Theta_3^*\xi_m(D)P_l^{-1}(D)$ with $\Theta_3^* = K_p^{-1}$ as

$$\Lambda(D)\Theta_3^*\xi_m(D)P_l^{-1}(D) = Q_l(D) + R_l(D)P_l^{-1}(D) \tag{245}$$

for some $M \times M$ polynomial matrices $Q_l(D)$ and $R_l(D)$ such that $\partial_{ci}[R_l(D)] < \partial_{ci}[P_l(D)] \leq \nu$. Then, we define $\Theta_1^*$, $\Theta_2^*$ and $\Theta_{20}^*$ from

$$\Theta_2^{*T} A(D) + \Theta_{20}^* \Lambda(D) = -R_l(D), \tag{246}$$

$$\Theta_1^{*T} A(D) = \Lambda(D) I_M - Q_l(D)Z_l(D) \tag{247}$$

(recall that $\partial[\Lambda(D)] = \nu - 1$ and $\partial[A(D)] = \nu - 2$ for the controller structure (9.102)). With this definition of $\Theta_2^*$, $\Theta_{20}^*$ and $\Theta_1^*$, (244) is satisfied, and so is (9.105).

From (9.105) (that is, (244)), we have the plant–model matching equation

$$I_M - \Theta_1^{*T} F(D) - (\Theta_2^{*T} F(D) + \Theta_{20}^*)G_0(D) = \Theta_3^* W_m^{-1}(D)G_0(D) \tag{248}$$

where $W_m(D) = \xi_m^{-1}(D)$. From this equation with $\lim_{D\to\infty} \Theta_3^* W_m^{-1}(D)G_0(D) = I_M$ as from the definition of $\xi_m(D)$, we have

$$\lim_{D\to\infty} \Theta_1^{*T} F(D) = 0, \tag{249}$$

which implies that $\partial[\Theta_1^{*T} A(D)] \leq \nu - 2$, that is, (247) is solvable.

# Plant Signal Identities for MIMO Cases

It has been shown that there exist $\Theta_1^*$, $\Theta_2^*$ and $\Theta_3^*$ to satisfy the matching equation

$$\Theta_1^{*T} A(D) P_0(D) + \Theta_2^{*T} A(D) Z_0(D) = \Lambda(D)(P_0(D) - \Theta_3^* \xi_m(D) Z_0(D)), \tag{250}$$

which is nominally (mathematically) equivalent to

$$I_M - \Theta_1^{*T} F(D) - \Theta_2^{*T} F(D) G_0(D) = \Theta_3^* W_m^{-1}(D) G_0(D). \tag{251}$$

Operating this identity on $u(t)$, we may get the plant signal identity:

$$u(t) - \Theta_1^{*T} \omega_1(t) - \Theta_2^{*T} \omega_2(t) = \Theta_3^* W_m^{-1}(D)[y](t). \tag{252}$$

(This identity was used in deriving (9.119), and its similar version with $\Theta_{20}^*$ was used in deriving (9.365).)

As an alternative procedure to obtain the plant parametrized signal identity (similar to that in (5.30) for the SISO case with $M = 1$), we start by considering (9.84) and (9.85):

$$\Theta_2^{*T} A(D) = Q_l(D) P_l(D) - \Lambda(D) K_p^{-1} \xi_m(D) \tag{253}$$

$$\Theta_1^{*T} A(D) = \Lambda(D) I_M - Q_l(D) Z_l(D). \tag{254}$$

and obtain the signal identities:

$$\Theta_2^{*T} \frac{A(D)}{\Lambda(D)}[y](t) = \frac{1}{\Lambda(D)} Q_l(D) P_l(D)[y](t) - K_p^{-1} \xi_m(D)[y](t) \tag{255}$$

$$\Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) = u(t) - \frac{1}{\Lambda(D)} Q_l(D) Z_l(D)[u](t). \tag{256}$$

Using the open-loop plant signal identity: $P_l(D)[y](t) = Z_l(D)[u](t)$, and from (255)–(256), we finally have the parametrized plant signal identity (252):

$$u(t) = \Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^{*T} \frac{A(D)}{\Lambda(D)}[y](t) + K_p^{-1} \xi_m(D)[y](t). \tag{257}$$

This identity holds for any input signal $u(t)$, in a feedback structure.

Similarly, for the controller structure (9.102) with nominal parameters:

$$u(t) = \Theta_1^{*T} \omega_1(t) + \Theta_2^{*T} \omega_2(t) + \Theta_{20}^* y(t) + \Theta_3^* r(t), \tag{258}$$

in which $\partial[\Lambda(D)] = \nu - 1$, we have the polynomial matching equation

$$\begin{aligned} &\Theta_1^{*T} A(D) P_0(D) + (\Theta_2^{*T} A(D) + \Theta_{20}^* \Lambda(D)) Z_0(D) \\ &= \Lambda(D)(P_0(D) - \Theta_3^* \xi_m(D) Z_0(D)) \end{aligned} \tag{259}$$

and the transfer matrix matching equation

$$I_M - \Theta_1^{*T} F(D) - (\Theta_2^{*T} F(D) + \Theta_{20}^*) G_0(D) = \Theta_3^* W_m^{-1}(D) G_0(D). \tag{260}$$

The nominal parameters $\Theta_1^*$, $\Theta_2^*$, $\Theta_{20}^*$ and $\Theta_3^*$ are defined from

$$(\Theta_2^{*T} A(D) + \Theta_{20}^* \Lambda(D)) = -R_l(D) = Q_l(D) P_l(D) - \Lambda(D) K_p^{-1} \xi_m(D) \tag{261}$$

$$\Theta_1^{*T} A(D) = \Lambda(D) I_M - Q_l(D) Z_l(D), \tag{262}$$

that is, dividing $\Lambda(D) K_p^{-1} \xi_m(D)$ on the right by $P_l(D)$ to get $R_l(D)$ and $Q_l(D)$. From these equations, we obtain the signal identities:

$$(\Theta_2^{*T} \frac{A(D)}{\Lambda(D)} + \Theta_{20}^*)[y](t) = \frac{1}{\Lambda(D)} Q_l(D) P_l(D)[y](t) - K_p^{-1} \xi_m(D)[y](t) \tag{263}$$

$$\Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) = u(t) - \frac{1}{\Lambda(D)} Q_l(D) Z_l(D)[u](t). \tag{264}$$

Using the open-loop plant signal identity: $P_l(D)[y](t) = Z_l(D)[u](t)$, we have the plant signal identity in a feedback form:

$$u(t) = \Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^{*T} \frac{A(D)}{\Lambda(D)}[y](t) + \Theta_{20}^* y(t) + K_p^{-1} \xi_m(D)[y](t) \tag{265}$$

which also holds for any input signal $u(t)$, similar to that in (257).

Both parametrized plant signal identities (257) and (265) are useful for adaptive control: either verify the nominal controller structure

$$u(t) = \Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^{*T} \frac{A(D)}{\Lambda(D)}[y](t) + \Theta_3^* r(t) \tag{266}$$

for the matching equation (250), or

$$u(t) = \Theta_1^{*T} \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^{*T} \frac{A(D)}{\Lambda(D)}[y](t) + \Theta_{20}^* y(t) + \Theta_3^* r(t) \tag{267}$$

for the matching equation (259), where $\Theta_3^* r(t) = K_p^{-1} \xi_m(D)[y_m](t)$ for the reference output $y_m(t) = \xi_m^{-1}(D)[r](t)$, leading to $\xi_m(D)[y - y_m](t) = 0$ exponentially, or they motivate the adaptive controller structure

$$u(t) = \Theta_1^T \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^T \frac{A(D)}{\Lambda(D)}[y](t) + \Theta_3 r(t) \tag{268}$$

for the matching equation (250), or

$$u(t) = \Theta_1^T \frac{A(D)}{\Lambda(D)}[u](t) + \Theta_2^T \frac{A(D)}{\Lambda(D)}[y](t) + \Theta_{20} y(t) + \Theta_3 r(t) \tag{269}$$

for the matching equation (259), leading to the desired tracking error equation:

$$e(t) = y(t) - y_m(t) = \xi_m^{-1}(D) K_p [\tilde{\Theta}^T \omega](t) \tag{270}$$

for $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ with

$$\Theta(t) = [\Theta_1^T(t), \Theta_2^T(t), \Theta_3(t)]^T, \ \Theta^* = [\Theta_1^{*T}, \Theta_2^{*T}, \Theta_3^*]^T \tag{271}$$

$$\omega(t) = [\omega_1^T(t), \omega_2^T(t), r^T(t)]^T \tag{272}$$

for the matching equation (250), or

$$\Theta(t) = [\Theta_1^T(t), \Theta_2^T(t), \Theta_{20}(t), \Theta_3(t)]^T, \ \Theta^* = [\Theta_1^{*T}, \Theta_2^{*T}, \Theta_{20}^*, \Theta_3^*]^T \tag{273}$$

$$\omega(t) = [\omega_1^T(t), \omega_2^T(t), y^T(t), r^T(t)]^T \tag{274}$$

for the matching equation (259). Both tracking error equations can be used to derive some desired estimation errors for designing stable adaptive laws to update the controller parameters $\Theta(t)$.

# Proof of $\lim_{t\to\infty} e(t) = 0$ in (9.187C)

To show $\lim_{t\to\infty} e(t) = 0$, we just need to prove that for any given $\delta > 0$, there exists a $T = T(\delta) > 0$ such that $\|e(t)\| < \delta$, for all $t > T$. For such a proof, we only require the existence of $T$ for each chosen $\delta$ which may be set to be arbitrarily small but do not consider the limit of $\delta$ going to zero, that is, $\delta > 0$ such that $\frac{1}{\delta}$ is finite.

In view of (9.187C) (for the continuous-time case), for any given $\delta > 0$, we can choose (think of) a pair of $K_3(s)$ and $H_3(s)$ to make $\frac{c_4}{a_3} < \frac{\delta}{2}$ (that is, $a_3 > \frac{2c_4}{\delta}$ is finite) (recall that $K_3(s)$ and $H_3(s)$ are not real but fictitious filters, and they are not employed in the control design but only used in the expression (decomposition) of $e(t)$ in (9.185C)), and let $T > 0$ such that $|z_1(t)| < \frac{\delta}{2}$ for any $t > T$ (such a $T = T(\delta)$ exists, because $\lim_{t\to\infty} z_1(t) = 0$; also note that $z_1(t)$ depends on $K_3(s)$). It follows that $\|e(t)\| < \delta$ for any $t > T$, which implies that $\lim_{t\to\infty} e(t) = 0$.

# Adaptive Robot Control for Time-Varying Parameters

Consider the manipulator dynamic equation (9.586):

$$D(q,t)\ddot{q} + \frac{\partial D(q,t)}{\partial t}\dot{q} + C(q,\dot{q},t)\dot{q} + \phi(q,t) = u. \tag{275}$$

With (9.590): $v = \dot{q}_d - \Lambda_0(q - q_d)$, $s = \dot{q} - v$, $e = q - q_d$, we have (9.592):

$$\begin{aligned} & D(q,t)\dot{s} + C(q,\dot{q},t)s \\ &= u - D(q,t)\dot{v} - C(q,\dot{q},t)v - \phi(q,t) - \frac{\partial D(q,t)}{\partial t}\dot{q} \\ &\stackrel{\triangle}{=} u - Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\theta^*(t) - \frac{\partial D(q,t)}{\partial t}\dot{q}, \end{aligned} \tag{276}$$

for some known function matrix $Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t) \in R^{n\times n_\theta}$ and unknown parameter vector $\theta^*(t) \in R^{n_\theta}$ which may be time-varying.

In this study, we consider the case when

$$\theta^*(t) = \theta_0^* + \delta\theta^*(t) \tag{277}$$

where $\theta_0^*$ is a constant vector and $\delta\theta^*(t)$ is the variation of $\theta^*(t)$ with respect to $\theta_0^*$ (note that both $\theta_0^*$ and $\delta\theta^*(t)$ are unknown). We will develop and analyze some alternative adaptive control schemes to that presented in (9.607)–(9.608).

### Adaptive Control Scheme I

As an alternative scheme to (9.607)–(9.608), we use the control law

$$u(t) = Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\theta_0(t) - m(t)\psi(t) - m_1(t)\psi_1(t) - K_D s(t), \tag{278}$$

$$m(t) = k_0\|\dot{q}(t) + v(t)\|f(q),\ k_0 > 0,\ \psi(t) = m(t)s(t), \tag{279}$$

$$m_1(t) = k_1\|Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\|,\ k_1 > 0,\ \psi_1(t) = m_1(t)s(t), \tag{280}$$

and the update law for the estimate $\theta_0(t)$ of $\theta_0^*$:

$$\dot{\theta}_0(t) = -\Gamma\left(Y^T(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)s + \sigma(t)\theta_0(t)\right),,\ \Gamma = \Gamma^T > 0 \tag{281}$$

where $\sigma(t)$ is a switching signal similar to that in (9.610), using a design parameter $\sigma_0 > 0$ and the knowledge of the upper bound $M_0$ on $\|\theta_0^*\|$:

$$\sigma(t) = \begin{cases} 0 & \text{if } \|\theta_0(t)\| < M_0, \\ \sigma_0\left(\frac{\|\theta_0(t)\|}{M_0} - 1\right) & \text{if } M_0 \le \|\theta_0(t)\| < 2M_0, \\ \sigma_0 & \text{if } \|\theta_0(t)\| \ge 2M_0. \end{cases} \tag{282}$$

This adaptive control scheme has the properties: all signals in the closed-loop system are bounded, and the tracking error $e(t) = q(t) - q_d(t)$ satisfies

$$\int_{t_1}^{t_2} \|e(t)\|^2 \, dt \leq \alpha_0 \left( \frac{\gamma^2}{k_0^2} + \frac{\gamma_1^2}{k_1^2} \right) (t_2 - t_1) + \beta_0 \tag{283}$$

for some constants $\alpha_0 > 0$, $\beta_0 > 0$ and any $t_2 > t_1 \geq 0$, where $\gamma_1 > 0$ is the upper bound on $\sup_{t \geq 0} \|\delta\theta^*(t)\|$. Moreover, $e(t) \in L^2$ and $\lim_{t\to\infty} e(t) = 0$ in the absence of parameter time variations, that is, when $\delta\theta^*(t) = 0$ and $\frac{\partial D(q,t)}{\partial t} = 0$.

The proof of these properties is based on the positive definite function

$$V_0(s, \tilde{\theta}_0) = \frac{1}{2}(s^T D s + \tilde{\theta}_0^T \Gamma^{-1} \tilde{\theta}_0),\ \tilde{\theta}_0(t) = \theta_0(t) - \theta_0^*,\ D = D(q(t),t), \tag{284}$$

which has the following time derivation:

$$\begin{aligned}
\dot{V}_0 = & -s^T(t) K_D s(t) - m^2(t) s^T(t) s(t) - m_1^2(t) s^T(t) s(t) - \sigma(t) \tilde{\theta}_0^T(t) \theta_0(t) \\
& - \frac{1}{2} s^T(t) \frac{\partial D(q,t)}{\partial t} (\dot{q}(t) + v(t)) - s^T(t) Y(q, q_d, \dot{q}, \dot{q}_d, \ddot{q}_d, t) \delta\theta^*(t) \\
\leq & -s^T(t) K_D s(t) - \left( m(t)\|s(t)\| - \frac{\gamma}{4k_0} \right)^2 + \frac{\gamma^2}{16k_0^2} \\
& - \left( m_1(t)\|s(t)\| - \frac{\gamma_1}{2k_1} \right)^2 + \frac{\gamma_1^2}{4k_1^2} - \sigma(t) \tilde{\theta}_0^T(t) \theta_0(t).
\end{aligned} \tag{285}$$

With this adaptive control scheme, as indicated by (283), the tracking performance can be influenced by the design parameters $k_0$ and $k_1$ in the feedback control law (278)–(280) (one may increase $k_0$ and $k_1$ to reduce the tracking error $e(t)$).

### Adaptive Control Scheme II

A different adaptive control scheme can be developed, employing a switching control law which uses adaptive estimates of parameter variation uncertainty bounds, to improve system tracking performance.

To derive such a scheme, we denote the parameter variation uncertainties as

$$g(q, \dot{q}, q_d, \dot{q}_d, t) = \frac{1}{2} \frac{\partial D(q,t)}{\partial t} (\dot{q} + v) = [g_1, g_2, \ldots, g_n]^T \tag{286}$$

$$h(q, \dot{q}, q_d, \dot{q}_d, \ddot{q}_d, t) = Y(q, q_d, \dot{q}, \dot{q}_d, \ddot{q}_d, t) \delta\theta^*(t) = [h_1, g_2, \ldots, h_n]^T \tag{287}$$

and make use of the bounding relationship

$$|g_i(q, \dot{q}, q_d, \dot{q}_d, t)| \leq a_i^* \alpha_i(q, \dot{q}, q_d, \dot{q}_d, t),\ i = 1, 2, \ldots, n \tag{288}$$

$$|h_i(q, \dot{q}, q_d, \dot{q}_d, \ddot{q}_d, t)| \leq b_i^* \beta_i(q, \dot{q}, q_d, \dot{q}_d, \ddot{q}_d, t),\ i = 1, 2, \ldots, n \tag{289}$$

for some unknown constants $a_i^*$ and $b_i^*$, and known functions $\alpha_i(q,\dot{q},q_d,\dot{q}_d,t)$ and $\beta_i(q,\dot{q},q_d,\dot{q}_d,\ddot{q}_d,t)$, $i=1,2,\ldots,n$.

If the parameters $a_i^*$ and $b_i^*$ were known, one could use the control law

$$u(t)=Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\theta_0(t)-\phi^*(t)-\phi_1^*(t)-K_Ds(t), \tag{290}$$

$$\phi^*(t)=[\mathrm{sgn}[s_1(t)]a_1^*\alpha_1,\mathrm{sgn}[s_2(t)]a_2^*\alpha_2,\ldots,\mathrm{sgn}[s_n(t)]a_n^*\alpha_n]^T, \tag{291}$$

$$\phi_1^*(t)=[\mathrm{sgn}[s_1(t)]b_1^*\beta_1,\mathrm{sgn}[s_2(t)]b_2^*\beta_2,\ldots,\mathrm{sgn}[s_n(t)]b_n^*\beta_n]^T, \tag{292}$$

where $\theta_0(t)$ is updated from (281), and the sgn function is defined as

$$\mathrm{sgn}[w]=\begin{cases}1 & \text{if } w>0,\\ 0 & \text{if } w=0,\\ -1 & \text{if } w<0.\end{cases} \tag{293}$$

Form $V_0$ defined in (284), this control law leads to

$$\begin{aligned}\dot{V}_0 &= -s^T(t)K_Ds(t)-s^T(t)\phi^*(t)-s^T(t)\phi_1^*(t)-\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t)\\ &\quad -\frac{1}{2}s^T(t)\frac{\partial D(q,t)}{\partial t}(\dot{q}(t)+v(t))-s^T(t)Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\delta\theta^*(t)\\ &= -s^T(t)K_Ds(t)-\sum_{i=1}^n|s_i(t)|a_i^*\alpha_i-\sum_{i=1}^n|s_i(t)|b_i^*\beta_i-\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t)\\ &\quad -\sum_{i=1}^n s_i(t)g_i-\sum_{i=1}^n s_i(t)h_i\\ &\le -s^T(t)K_Ds(t).\end{aligned} \tag{294}$$

The last equality follows from the facts that $\sum_{i=1}^n|s_i(t)|a_i^*\alpha_i-\sum_{i=1}^n s_i(t)g_i\ge 0$, $\sum_{i=1}^n|s_i(t)|b_i^*\beta_i-\sum_{i=1}^n s_i(t)h_i\ge 0$ and $\sigma(t)\tilde{\theta}_0^T(t)\theta_0(t)\ge 0$. From (294), one may conclude that all signals in the closed-loop system are bounded, and the tracking error $e(t)=q(t)-q_d(t)$ converges to zero as $t$ goes to $\infty$.

When the parameters $a_i^*$ and $b_i^*$ are unknown, one can use the control law

$$u(t)=Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\theta_0(t)-\phi(t)-\phi_1(t)-K_Ds(t), \tag{295}$$

$$\phi(t)=[\mathrm{sgn}[s_1(t)]a_1(t)\alpha_1,\mathrm{sgn}[s_2(t)]a_2(t)\alpha_2,\ldots,\mathrm{sgn}[s_n(t)]a_n(t)\alpha_n]^T, \tag{296}$$

$$\phi_1(t)=[\mathrm{sgn}[s_1(t)]b_1(t)\beta_1,\mathrm{sgn}[s_2(t)]b_2(t)\beta_2,\ldots,\mathrm{sgn}[s_n(t)]b_n(t)\beta_n]^T, \tag{297}$$

where $\theta_0(t)$ is updated from (281), and the parameters $a_i(t)$ and $b_i(t)$ are estimates of $a_i^*$ and $b_i^*$ and updated from the adaptive laws:

$$\dot{a}_i(t)=\kappa_{ai}|s_i(t)|\alpha_i(q,\dot{q},q_d,\dot{q}_d,t),\ \kappa_{ai}>0,\ i=1,2,\ldots,n, \tag{298}$$

$$\dot{b}_i(t)=\kappa_{bi}|s_i(t)|\beta_i(q,\dot{q},q_d,\dot{q}_d,\ddot{q}_d,t),\ \kappa_{bi}>0,\ i=1,2,\ldots,n. \tag{299}$$

Consider the positive definite function

$$V(s,\tilde{\theta}_0,\tilde{a}_i,\tilde{b}_i) = \frac{1}{2}(s^T D s + \tilde{\theta}_0^T \Gamma^{-1}\tilde{\theta}_0 + \sum_{i=1}^{n} \kappa_{ai}^{-1}\tilde{a}_i^2 + \sum_{i=1}^{n} \kappa_{bi}^{-1}\tilde{b}_i^2), \tag{300}$$

where $\tilde{\theta}_0(t) = \theta_0(t) - \theta_0^*$, $D = D(q(t),t)$, $\tilde{a}_i(t) = a_i(t) - a_i^*$, $\tilde{b}_i(t) = b_i(t) - b_i^*$, $i = 1,2,\ldots,n$. Using (276), (295), (281), (298) and (299), we have

$$\begin{aligned}
\dot{V} &= -s^T(t)K_D s(t) - s^T(t)\phi(t) - s^T(t)\phi_1(t) - \sigma(t)\tilde{\theta}_0^T(t)\theta_0(t) \\
&\quad -\frac{1}{2}s^T(t)\frac{\partial D(q,t)}{\partial t}(\dot{q}(t)+v(t)) - s^T(t)Y(q,q_d,\dot{q},\dot{q}_d,\ddot{q}_d,t)\delta\theta^*(t) \\
&\quad + \sum_{i=1}\tilde{a}_i\kappa_{ai}^{-1}\dot{a}_i + \sum_{i=1}\tilde{b}_i\kappa_{bi}^{-1}\dot{b}_i \\
&= -s^T(t)K_D s(t) - \sum_{i=1}^{n}|s_i(t)|a_i(t)\alpha_i - \sum_{i=1}^{n}|s_i(t)|b_i(t)\beta_i - \sigma(t)\tilde{\theta}_0^T(t)\theta_0(t) \\
&\quad - \sum_{i=1}^{n} s_i(t)g_i - \sum_{i=1}^{n} s_i(t)h_i + \sum_{i=1}\tilde{a}_i\kappa_{ai}^{-1}\dot{a}_i + \sum_{i=1}\tilde{b}_i\kappa_{bi}^{-1}\dot{b}_i \\
&\le -s^T(t)K_D s(t) - \sum_{i=1}^{n}|s_i(t)|a_i(t)\alpha_i - \sum_{i=1}^{n}|s_i(t)|b_i(t)\beta_i - \sigma(t)\tilde{\theta}_0^T(t)\theta_0(t) \\
&\quad + \sum_{i=1}^{n}|s_i(t)|a_i^*\alpha_i + \sum_{i=1}^{n}|s_i(t)|b_i^*\beta_i + \sum_{i=1}\tilde{a}_i\kappa_{ai}^{-1}\dot{a}_i + \sum_{i=1}\tilde{b}_i\kappa_{bi}^{-1}\dot{b}_i \\
&= -s^T(t)K_D s(t) - \sigma(t)\tilde{\theta}_0^T(t)\theta_0(t) \\
&\le -s^T(t)K_D s(t).
\end{aligned} \tag{301}$$

This result also implies that all signals in the closed-loop system are bounded, and the tracking error $e(t) = q(t) - q_d(t)$ converges to zero as $t$ goes to $\infty$.

However, since the adaptive control scheme (295) uses the sgn functions in $\phi(t)$ and $\phi_1(t)$ and such switching signals are discontinuous when $s_i(t)$ passes through zero, it may lead to chattering of system response.

**Remark 5** The adaptive control scheme (295) may have certain advantage for performance even if when the parameters $a_i^*$ and $b_i^*$ are known. This is because the parameters $a_i^*$ and $b_i^*$ are only the upper bounds for the parameter variation uncertainties $g_i$ and $h_i$ in (286) and (287), and some smaller (and unknown) bounds may exist and can be estimated by the adaptive laws (298) and (299). The use of smaller bounds is desirable because it leads to smaller control signals. In this case, the adaptive laws (298) and (299) can be modified by setting

$$\dot{a}_i(t) = 0,\ t \ge \tau \ \text{ if } a_i(\tau) = a_i^*, \tag{302}$$

$$\dot{b}_i(t) = 0,\ t \ge \tau \ \text{ if } b_i(\tau) = b_i^* \tag{303}$$

With this modification, we also have $\dot{V} \leq -s^T(t)K_D s(t)$, as desired. $\square$

**Adaptive Control Scheme III**

As mentioned above the use of the discontinuous sgn functions in $\phi(t)$ and $\phi_1(t)$ in the adaptive control scheme (295) may cause chattering of system response. To overcome possible chattering, we can modify the control law (295) as

$$u(t) = Y(q, q_d, \dot{q}, \dot{q}_d, \ddot{q}_d, t)\theta_0(t) - \hat{\phi}(t) - \hat{\phi}_1(t) - K_D s(t), \tag{304}$$

$$\hat{\phi}(t) = [\text{sat}[s_1(t); \varepsilon_1]a_1(t)\alpha_1, \ldots, \text{sat}[s_n(t); \varepsilon_n]a_n(t)\alpha_n]^T, \tag{305}$$

$$\hat{\phi}_1(t) = [\text{sat}[s_1(t); \eta_1]b_1(t)\beta_1, \ldots, \text{sat}[s_n(t); \eta_n]b_n(t)\beta_n]^T, \tag{306}$$

where the sat function is defined as

$$\text{sat}[s_i; x_i] = \begin{cases} 1 & \text{if } s_i > x_i, \\ \frac{s_i}{x_i} & \text{if } |s_i| \leq x_i, \\ -1 & \text{if } s_i < -x_i \end{cases} \tag{307}$$

for some chosen $x_i > 0$ (for $x_i = \varepsilon_i$ or $x_i = \eta_i$ in (305) and (306)), $i = 1, 2, \ldots, n$, with the associated indicator functions

$$\chi[s_i; x_i] = \begin{cases} 1 & \text{if } |s_i| > x_i, \\ 0 & \text{if } |s_i| \leq x_i. \end{cases} \tag{308}$$

Such functions have the property: $\chi[s_i; x_i](1 - \chi[s_i; x_i]) = 0$ (that is, $\chi[s_i; x_i] = 0$ whenever $1 - \chi[s_i; x_i] = 1$, and $\chi[s_i; x_i] = 1$ whenever $1 - \chi[s_i; x_i] = 0$).

The adaptive laws are also modified as

$$\dot{a}_i(t) = \chi[s_i; \varepsilon_i]\kappa_{ai}|s_i(t)|\alpha_i(q, \dot{q}, q_d, \dot{q}_d, t),\ \kappa_{ai} > 0,\ i = 1, 2, \ldots, n, \tag{309}$$

$$\dot{b}_i(t) = \chi[s_i; \eta_i]\kappa_{bi}|s_i(t)|\beta_i(q, \dot{q}, q_d, \dot{q}_d, \ddot{q}_d, t),\ \kappa_{bi} > 0,\ i = 1, 2, \ldots, n. \tag{310}$$

With this modification, we have

$$\begin{aligned}
\dot{V} &= -s^T(t)K_D s(t) - s^T(t)\hat{\phi}(t) - s^T(t)\hat{\phi}_1(t) - \sigma(t)\tilde{\theta}_0^T(t)\theta_0(t) \\
&\quad - \frac{1}{2}s^T(t)\frac{\partial D(q,t)}{\partial t}(\dot{q}(t) + v(t)) - s^T(t)Y(q, q_d, \dot{q}, \dot{q}_d, \ddot{q}_d, t)\delta\theta^*(t) \\
&\quad + \sum_{i=1} \tilde{a}_i \kappa_{ai}^{-1}\dot{a}_i + \sum_{i=1} \tilde{b}_i \kappa_{bi}^{-1}\dot{b}_i \\
&= -s^T(t)K_D s(t) - \sum_{i=1}^{n} s_i(t)\text{sat}[s_i; \varepsilon_i]a_i(t)\alpha_i - \sum_{i=1}^{n} s_i(t)\text{sat}[s_i; \eta_i]b_i(t)\beta_i
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{n} s_i(t) g_i - \sum_{i=1}^{n} s_i(t) h_i + \sum_{i=1} \tilde{a}_i \kappa_{ai}^{-1} \dot{a}_i + \sum_{i=1} \tilde{b}_i \kappa_{bi}^{-1} \dot{b}_i - \sigma(t) \tilde{\theta}_0^T(t) \theta_0(t) \\
\leq\, & -s^T(t) K_D s(t) - \sum_{i=1}^{n} s_i(t) \mathrm{sat}[s_i; \varepsilon_i] a_i(t) \alpha_i - \sum_{i=1}^{n} s_i(t) \mathrm{sat}[s_i; \eta_i] b_i(t) \beta_i \\
&+ \sum_{i=1}^{n} |s_i(t)| a_i^* \alpha_i + \sum_{i=1}^{n} |s_i(t)| b_i^* \beta_i + \sum_{i=1} \tilde{a}_i \kappa_{ai}^{-1} \dot{a}_i + \sum_{i=1} \tilde{b}_i \kappa_{bi}^{-1} \dot{b}_i - \sigma(t) \tilde{\theta}_0^T(t) \theta_0(t) \\
=\, & -s^T(t) K_D s(t) - \sum_{i=1}^{n} (1 - \chi[s_i; \varepsilon_i]) s_i(t) \mathrm{sat}[s_i; \varepsilon_i] a_i(t) \alpha_i \\
&- \sum_{i=1}^{n} (1 - \chi[s_i; \eta_i]) s_i(t) \mathrm{sat}[s_i; \eta_i] b_i(t) \beta_i + \sum_{i=1}^{n} (1 - \chi[s_i; \varepsilon_i]) |s_i(t)| a_i^* \alpha_i \\
&+ \sum_{i=1}^{n} (1 - \chi[s_i; \eta_i]) |s_i(t)| b_i^* \beta_i - \sigma(t) \tilde{\theta}_0^T(t) \theta_0(t).
\end{aligned}
\tag{311}
$$

This modified scheme would ensure the closed-loop signal boundedness but not the asymptotic convergence of the tracking error $e(t) = q(t) - q_d(t)$ to zero (only a bounded tracking error $e(t) = q(t) - q_d(t)$ of the order $\varepsilon_i$ and $\eta_i$).

# Derivation of (10.152)

In this case, the expression (5.30) also holds

$$u(t) = \phi_1^{*T} \frac{a(s)}{\Lambda(s)}[u](t) + \phi_2^{*T} \frac{a(s)}{\Lambda(s)}[y](t) + \phi_{20}^* y(t) + \phi_3^* P_m(s)[y](t) \tag{312}$$

(with $\theta_i^*$ replaced by $\phi_i^*$ as the new notation and the exponentially decaying term $\epsilon_1(t)$ ignored). Recall (10.39):

$$u(t) = u_d(t) + (\theta - \theta^*)^T \omega(t) + d_N(t) \tag{313}$$

and (10.15) (with $a_s(t) = 0$ for simplicity):

$$u_d(t) = -\theta^T(t)\omega(t) \tag{314}$$

from which we have

$$u(t) = -\theta^{*T} \omega(t) + d_N(t). \tag{315}$$

Using (313) for $u(t)$ in the left side of (312) and (315) for $u(t)$ in the right side of (312), we obtain

$$\begin{aligned} & u_d(t) + (\theta - \theta^*)^T \omega(t) + d_N(t) \\ = & \phi_1^{*T} \frac{a(s)}{\Lambda(s)}[-\theta^{*T} \omega + d_N](t) + \phi_2^{*T} \frac{a(s)}{\Lambda(s)}[y](t) + \phi_{20}^* y(t) + \phi_3^* P_m(s)[y](t). \end{aligned} \tag{316}$$

Subtracting (316) from (10.151) and recalling (10.148), we have (10.152).

# Backstepping Design with Nonsmooth Inverse Signal

For an adaptive inverse control scheme using a backstepping feedback design for the output-feedback nonlinear systems (see Section 10.6.3), the nonlinearity inverse signal $\omega(t)$ in (10.280), defined in (10.21) or (10.35) for the case of an adaptive dead-zone inverse $\widehat{DI}(\cdot)$ or an adaptive backlash inverse $\widehat{BI}(\cdot)$, is not smooth (it is not even continuous; but it is well-defined). A backstepping-based feedback control design needs certain smoothness condition. Will the nonsmoothness of $\omega(t)$ cause any problem? The answer is "no". It turns out that the use of filters in the state observer helps to avoid the potential difficulty with nonsmoothness of such signals in a backstepping design.

Starting from (10.277), (10.283) and (10.284), we see that the nonsmooth signal $\omega(t)$ passes the filters $\frac{p_{ij}(s)}{\Lambda(s)}$, where $\frac{p_{ij}(s)}{\Lambda(s)}$ are the components of $(sI - A_0)^{-1}$ for $A_0 = A - kc$ with $k = [k_1, k_2, \ldots, k_n]^T$ and $(A,\, c)$ in (10.272). This means that only the signals $\omega_{ij}(t)$ in (10.285) are associated with the nonsmooth signal $\omega(t)$. In fact, it is the signal $\hat{x}_2(t)$ defined in (10.128) that plays a key role in the backstepping design procedure, that is, only those of the signals $\omega_{ij}(t)$, for $i = 0, 1, \ldots, m$ and $j = 2$, are crucial in the backstepping design.

From the canonical form of $A$ and $c$, we can obtain

$$p_{02}(s) = s + k_1,\ p_{12}(s) = s(s + k_1),\ p_{22}(s) = s^2(s + k_1),\ \ldots,$$
$$p_{n-2,2}(s) = s^{n-2}(s + k_1),\ p_{n-1,2}(s) = -k_2 s^{n-2} - k_3 s^{n-3} - \cdots - k_{n-1} s - k_n. \quad (317)$$

This calculation can be verified by using the symbolic algebra operations in Matlab. For example, for $n = 5$, we can use

```
a=sym('[s+k1, -1, 0, 0, 0; k2, s, -1, 0, 0;
k3, 0, s, -1, 0; k4, 0, 0, s, -1; k5, 0, 0, 0, s]')
b=inv(a)
c=symmul(det(a),b)
```

to get the numerator matrix of $(sI - A_0)^{-1}$ as

```
[s^4,                          s^3,                s^2,
                                  s,                            1]
[-s^3*k2-k3*s^2-k4*s-k5, s^3*(s+k1),         s^2*(s+k1),
                           (s+k1)*s,                     s+k1]
[-s*(k3*s^2+k4*s+k5),-k3*s^2-k4*s-k5,s^2*(k2+s^2+s*k1),
                    s*(k2+s^2+s*k1),         k2+s^2+s*k1]
[-s^2*(k4*s+k5),           -s*(k4*s+k5),      -k4*s-k5,
              s*(k2*s+k3+s^3+s^2*k1), k2*s+k3+s^3+s^2*k1]
```

```
[-k5*s^3,                    -k5*s^2,         -k5*s,
     -k5,                        s^4+k1*s^3+k2*s^2+k3*s+k4]
```

In the backstepping design procedure, the derivatives $\omega_{i2}^{(j)}(t)$, $i = 0,1,\ldots,m$, $j = 0,1,\ldots,\rho-1$, are needed, where $\rho = n-m$. For the case of $\rho = n-m = 1$, no derivative of $\omega_{i2}(t)$ is used. For $\rho = n-m \geq 2$, the highest degree of $s^{\rho-1}p_{i2}(s)$, $i = 0,1,\ldots,m-1$, is $\rho-1+m = n-1$, and the degree of $s^{\rho-1}p_{m2}(s)$ is $n$, so that $\frac{s^{\rho-1}p_{i2}(s)}{\Lambda(s)}$, $i = 0,1,\ldots,m-1$, are strictly proper, and $\frac{s^{\rho-1}p_{m2}(s)}{\Lambda(s)}$ is proper. Therefore, the signals $\omega_{i2}^{(j)}(t)$, $i = 0,1,\ldots,m$, $j = 0,1,\ldots,\rho-1$, are well-defined, and so all other signals related the inverse signal $\omega(t)$ in the backstepping design procedure, that is, the nonsmoothness $\omega(t)$ does not cause any problem for the backstepping-based adaptive inverse control scheme.