Adaptive Control of Systems with Backlash*

GANG TAO† and PETAR V. KOKOTOVIĆ†

For systems with unknown backlash an adaptive inverse controller promises to significantly improve system performance.

Key Words—Adaptive backlash inverse; adaptive control; backlash; control system design; nonlinear systems; parameter estimation; stability.

Abstract—We develop a mathematical model of backlash inverse and give a parametrization of the error caused by its estimate. We then design an adaptive backlash inverse controller for unknown plants with backlash and prove the global boundedness of the closed-loop signals. Simulations show major improvements of system performance.

1. INTRODUCTION

System components, such as hydraulic servo-valves or gears often have dead-zone, hysteresis or backlash characteristics (Krasnosel'ski and Pokrovskii, 1983; Mayergoyz, 1991; Netushil, 1973; Truxal, 1958). These nondifferentiable nonlinearities severely limit overall system performance. Their parameters are often unknown, and an appropriate control strategy for such systems is adaptive. However most adaptive control results are for linear plants or plants with differentiable nonlinearities and are not applicable to nondifferentiable nonlinearities. This situation motivated a research program initiated by two recent papers which deal with dead-zone at the input of a linear part of the plant (Recker et al., 1991; Tao and Kokotovic, 1991). In this paper we continue this research direction and solve a more difficult problem with backlash. We propose an adaptive backlash inverse design which guarantees signal boundedness and results in major improvements of system performance.

We consider a plant with a linear part \( G(D) \) and a backlash nonlinearity \( B(\cdot) \) at its input:

\[
y(t) = G(D)[u(t)], \quad u(t) = B(\nu(t)),
\]

where \( D \) is used to denote, as the case may be, the Laplace transform variable or the differential operator \( D[x](t) = \dot{x}(t) \) in continuous time, and the \( z \)-transform variable or the advance operator \( D[x](k) = x(k + 1) \) in discrete time. Both \( G(D) \) and \( B(\cdot) \) are unknown. Only the plant output \( y(t) \) is measured and the accessible control input is \( \nu(t) \). The backlash output \( u(t) \) is not accessible.

We first develop a backlash inverse for the continuous-time case and derive its discrete-time counterpart. We then design an adaptive control scheme for the discrete-time version of the unknown plant (1.1). In Section 2 we introduce the concept of a backlash inverse \( BI(\cdot) \) and investigate its implementation. If the backlash characteristic \( B(\cdot) \) were known, then an exact backlash inverse \( BI(\cdot) \) inserted between a linear controller and the plant would cancel the effect of backlash. In most applications, the backlash nonlinearity cannot be accurately modeled and only an estimate of its inverse can be implemented. In Section 3, we derive a linear-like parametrization convenient for adaptive control. In Section 4, we use a first-order example to introduce the adaptive backlash inverse design and illustrate performance improvement achieved. Our main result is given in Section 5 where we design the adaptive backlash inverse system for unknown plants of higher relative degree.

2. BACKLASH AND ITS RIGHT INVERSE

In this section we develop a backlash inverse \( BI(\cdot) \) and present its implementation to be cascaded with the backlash \( B(\cdot) \) at the input of the plant.

2.1. Backlash characteristic

The backlash characteristic \( B(\cdot) \) with input \( \nu(t) \) and output \( u(t) \) is described by two straight lines, upward and downward sides of \( B(\cdot) \), connected with horizontal line segments. The upward side of \( B(\cdot) \) is active when both \( \nu(t) \) and...
$u(t)$ grow:

$$u(t) = m_r(v(t) - c_r), \ m_r > 0, \ \text{and} \ \dot{v}(t) > 0.$$  
(2.1)

If $\dot{v}(t)$ changes sign at $t = t_1$ and $\dot{v}(t) < 0$ for $t > t_1$, then the upward side becomes inactive and the motion for $t > t_1$ proceeds along the inner segment where

$$u(t) = u(t_1) = \text{constant.}$$  
(2.2)

The downward side of $B(\cdot)$ is

$$u(t) = m_l(v(t) - c_l), \ m_l > 0, \ c_l < c_r, \ \text{and} \ \dot{v}(t) < 0.$$  
(2.3)

It is reached at $t = t_2 > t_1$ when

$$u(t_1) = m_l(v(t_2) - c_l).$$  
(2.4)

During the motion on the inner segment with $u(t) = u(t_1)$, the following condition is satisfied:

$$\frac{u(t_1)}{m_l} + c_l < v(t) < u(t_1) + c_r.$$  
(2.5)

Starting with $t = t_2$, the downward side continues to be active as long as $\dot{v}(t) < 0$. If at $t = t_3 > t_2$ $\dot{v}(t)$ changes its sign again, and $\dot{v}(t) > 0$ for $t > t_3$, then the downward side becomes inactive and the motion proceeds on the inner segment where

$$u(t) = u(t_3) = \text{constant},$$  
(2.6)

until at $t = t_4 > t_3$, the upward side becomes active, where $t_4$ satisfies

$$u(t_3) = m_r(v(t_4) - c_r).$$  
(2.7)

This narrative description of the backlash characteristic is mathematically modeled by:

$$\dot{u}(t) = \begin{cases} 
m_r \dot{v}(t) & \text{if } \dot{v}(t) > 0 \text{ and } u(t) = m_r(v(t) - c_r), \\
m_l \dot{v}(t) & \text{if } \dot{v}(t) < 0 \text{ and } u(t) = m_l(v(t) - c_l), \\
0 & \text{if } m_l(v(t) - c_l) < u(t) < m_r(v(t) - c_r), \\
& \text{of } \dot{v}(t) > 0 \text{ and } u(t) = m_r(v(t) - c_r), \\
& \text{or } \dot{v}(t) < 0 \text{ and } u(t) = m_l(v(t) - c_l), \\
& \text{or } \dot{v}(t) = 0. 
\end{cases}$$  
(2.8)

Although this model allows the upward and downward slopes to be different $m_r \neq m_l$, provided that the intersection of the two lines is not in the region of practical interest, we only consider the usual backlash characteristic with two parallel sides $m_r = m_l = m$.

A typical motion on such a characteristic initiated at $t = 0$ with $v(0) = 0$ and $u(0) = 0$ is shown in Fig. 1, where $v(t)$ and $u(t)$ are plotted along two synchronized orthogonal $t$-axes.

It is useful to treat the model (2.8) as a first-order dynamical system and consider $u(t)$ as its state. With an initial condition $u(0)$, the knowledge of $v(t)$ and $\dot{v}(t)$ uniquely defines $u(t)$ for $t \geq 0$. We will restrict $u(t)$ to be piecewise continuous. Except at points of discontinuity of $v(t)$, its derivative $\dot{v}(t)$ will also be piecewise continuous. We note that $u(t)$ is "more discontinuous" than $v(t)$. For example, even if $v(t)$ is twice differentiable, $\dot{u}(t)$ may still be only piecewise continuous, due to the inner segments on which $\dot{u}(t) = 0$ even though $\dot{v}(t) \neq 0$.

Another important observation concerns the discontinuity of $u(t)$ caused by an inconsistent initialization when the pair $v(t), u(t)$ is not a point on the backlash characteristic in the $(v, u)$-plane. After a jump in $u(t)$ the pair $v(t^\ast), u(0^\ast)$ will uniquely define a point on the backlash characteristic and $v(t), u(t)$ will remain on it thereafter.

Because of the dependence of $\dot{u}(t)$ not only on $v(t)$ and $u(t)$, but also on $\dot{v}(t)$, we think of the input-output mapping from $v(t)$ to $u(t)$ defined by (2.8) as a description of a "relative degree zero" system having a causal right inverse. Next we develop such an inverse and use it as a part of our new controller structure.

2.2. Backlash inverse

The most damaging effect of backlash on system performance is the delay corresponding to the time needed to traverse an inner segment of $B(\cdot)$. The ideal backlash inverse $B^{-1}(\cdot)$ will make the traverse of this segment instantaneous and thus cancel this undesirable backlash effect. Another undesirable effect of backlash is the information loss occurring on an inner segment when the output $u(t)$ remains constant while the input $v(t)$ continues to change (see the "chopped" $u(t)$ in Fig. 1). These two undesirable effects are eliminated with the backlash inverse $B^{-1}(\cdot)$ defined by the following mapping from $u_d(t)$ to $v(t)$:

$$\dot{v}(t) = \begin{cases} 
\frac{1}{m_r} \dot{u}_d(t) & \text{if } \dot{u}_d(t) > 0 \text{ and } v(t) = u_d(t) + c_r, \\
\frac{1}{m_l} \dot{u}_d(t) & \text{if } \dot{u}_d(t) < 0 \text{ and } v(t) = u_d(t) + c_l, \\
0 & \text{if } \dot{u}_d(t) = 0 
\end{cases}$$  
(2.9)

$$g(t, t) \dot{u}_d(t) + \dot{v}(t) = \frac{u_d(t)}{m_r} + c_r, \ \text{if } \dot{u}_d(t) > 0 \text{ and } v(t) = u_d(t) + c_r,$$

$$-g(t, t) \dot{u}_d(t) + \dot{v}(t) = \frac{u_d(t)}{m_l} + c_l, \ \text{if } \dot{u}_d(t) < 0 \text{ and } v(t) = u_d(t) + c_l.$$  

In this definition, the inverse of a horizontal...
segment of the backlash characteristic is a vertical jump defined as the time integral of the impulse:

\[ g(t, \tau) = \delta(t - \tau) \left( \frac{1}{m_r} - \frac{1}{m_l} \right) u_d(\tau) + c_r - c_l, \]

where \( \delta(t) \) is the Dirac \( \delta \)-function. Thus an upward jump in the backlash inverse is

\[ v(t^+) = v(t^-) + \int_{t^-}^{t^+} g(\tau, t) \, d\tau = v(t^-) \]

\[ + \left( \frac{1}{m_r} - \frac{1}{m_l} \right) u_d(t^-) + c_r - c_l \]

\[ = \frac{u_d(t^-)}{m_r} + c_r. \]

The effect of this jump in \( BI(\cdot) \) will be to eliminate the delay caused by a segment in \( B(\cdot) \).

In a similar manner the use of (2.9) restores the information that would have been lost in (2.8).

We show this by proving that \( BI(\cdot) \) defined in (2.9) is the right inverse of \( B(\cdot) \) defined by (2.8).

Lemma 2.1 (Backlash inverse). The characteristic \( BI(\cdot) \) defined by (2.9) is the right inverse of the characteristic \( B(\cdot) \) defined by (2.8) in the sense that

\[ Ud(t_0) = B(BI(u_d(t_0))) = B(BI(u_d(t))) = Ud(t), \]

for any piecewise continuous \( u_d(t) \) and any \( t_0 \geq 0 \).

Proof. Suppose that \( u_d(t) > 0 \) for \( t \in [t_0, t_1] \) and some \( t_1 > t_0 \). First, if \( v(t_0) = u_d(t_0)/m_r + c_r \) and \( u(t_0) = m_r(v(t_0) - c_r) \), then it follows from (2.9), (2.8) that \( \dot{u}(t) = m_r \ddot{v}(t) = m_r(\dot{u}_d(t)/m_r) = \dot{u}_d(t) \) for \( t \in [t_0, t_1] \) with \( u(t_0) = u_d(t_0) \). Hence \( B(BI(u_d(t))) = u_d(t) \) for any \( t \in [t_0, t_1] \). Second, if \( v(t_0) = u_d(t_0)/m_l + c_l \) and \( u(t_0) = m_l(v(t_0) - c_l) \), then, according to (2.9), \( v(t) \) will have a jump at \( t = t_0 \) so that \( v(t) = u_d(t)/m_r + c_r \) for \( t \geq t_0 \). The
jump in \( v(t) \) makes \( u(t) \) traverse an inner segment so that \( u(t_0^+) = m_r(v(t_0^-) - c_r) \), which reduces to the first case above.

When \( \dot{u}_d(t) < 0 \) for \( t \in [t_0, t_1] \), a similar analysis shows that \( B(BI(\dot{u}_d(t))) = u_d(t) \) for any \( t \in [t_0, t_1] \). If \( \dot{u}_d(t) = 0 \) for \( t \in [t_0, t_1] \), then \( B(BI(\dot{u}_d(t))) = u_d(t) \) holds for any \( t \in [t_0, t_1] \).

If \( \dot{u}_d(t) \) changes the sign at \( t = t_1 \), then we can repeat the procedure, and show that \( B(BI(\dot{u}_d(t))) = u_d(t) \) for any \( t \geq t_0 \). \( \nabla \)

The mapping (2.9)-(2.11) may not define a backlash inverse only if the signal \( u_d(t) \) is such that \( v(t) \) and \( u(t) \) never leave an inner segment. This situation can happen only if \( v(0), u(0) \) are initially on an inner segment and \( \dot{u}_d(t) = 0 \) for \( t \geq 0 \) or if \( \dot{u}_d(t) \) does not change sign but the total increment of \( u_d(t)/m_r \) (or decrement of \( u_d(t)/m_t \)) is insufficient for \( v(t), u(t) \) to leave the segment.

As \( u_d(t) \) is the design signal at our disposal, the above situation can be remedied. If \( u_d(t) \) does not reach \( t_0 \) defined in (2.12), then \( u_d(t) \), \( v(t) \) are initialized by

\[
v(t_0^+) = \begin{cases} 
u_d(t_0) - c_r & \text{if } v(t_0) = \frac{u_d(t_0)}{m_t} + c_t \geq 0 \\ 
u_d(t_0) + c_t & \text{if } v(t_0) = \frac{u_d(t_0)}{m_t} - c_t \leq 0 \end{cases}
\] (2.13)

This will always result in \( u_d(t_0^+) = B(BI(\dot{u}_d(t_0^+))) \) and (2.12) holds thereafter.

When the exact backlash parameters \( m_r, m_t, c_r, c_t \) are unknown, we will use the estimated backlash parameters to design an adaptive backlash inverse.

Let \( \hat{m}_r(t), \hat{c}_r(t), \hat{m}_t(t), \hat{c}_t(t) \) be estimates of \( m_r, m_t, c_r, c_t \), and denote the adaptive backlash inverse \( BI(\cdot) \) as \( BI(\hat{m}_r(t), \hat{c}_r(t), \hat{m}_t(t), \hat{c}_t(t), \; \cdot) \).

Graphically, the backlash inverse (2.9)-(2.11) is depicted in Fig. 2 by two straight lines and instantaneous vertical transitions between the lines, where the downward side is

\[
v(t) = \frac{u_d(t)}{m_r(t)} - \hat{c}_r(t), \quad \text{and} \quad \dot{u}_d(t) < 0, \quad (2.14)
\]

and the upward side is

\[
v(t) = \frac{u_d(t)}{m_t(t)} + \hat{c}_t(t), \quad \text{and} \quad \dot{u}_d(t) > 0. \quad (2.15)
\]

Instantaneous vertical transitions take place whenever \( \dot{u}_d(t) \) changes its sign. On the lines \( v(t) = 0 \) whenever \( \dot{u}_d(t) = 0 \).

In Fig. 2, the motion of \( v(t) \), \( u_d(t) \) starts with \( u_d(0) = v(0) = 0 \). At \( t = 0 \), \( v(t) \) vertically moves to the upward side. For \( t \in (0, t_1] \), \( \dot{u}_d(t) > 0 \) does not change sign so the motion stays on the
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upward side (2.15). For \( t \in (t_1, t_2) \), \( \dot{u}_d(t) < 0 \), the downward side (2.14) is active. At \( t = t_1 \), the sign of \( \dot{u}_d(t) \) changes causing an instantaneous vertical transition from the upward side to the downward side. Because of the subsequent sign changes of \( \dot{u}_d(t) \), two more instantaneous vertical transitions take place; one upward at \( t = t_2 \) and another one downward at \( t = t_3 \). The magnitude of the vertical translation of \( v(t) \) is equal to the length of the estimated inner segment of \( B(-) \). If an exact backlash inverse is used, that is, if \( m_t = m_r \), \( c_t = c_r \), \( m_r = m_r \), and \( G(-) = c_r \), then, after initialization of the backlash inverse, the backlash output \( u(t) \) is equal to \( u_d(t) \), that is, \( u(t) = B(B(u_d(t))) = u_d(t) \).

2.3. Discrete-time representation

In most applications the accessible control \( v(t) \) is piecewise constant, i.e. \( v(t) = v(t_k) \) for \( t \in [t_k, t_{k+1}) \), \( k = 0, 1, 2, \ldots \). For such discrete-time applications the backlash model (2.8) is not appropriate because of the discontinuity of signals. However, from the same physical description of backlash given by (2.1)-(2.7), we can obtain the following discrete-time backlash model \( u(k) = B(v(k)) \):

\[
u(k) = \begin{cases} 
  m_t(v(k) - c_t) & \text{for } v(k) \leq v_l \\
  m_r(v(k) - c_r) & \text{for } v(k) \geq v_r \\
  u(k-1) & \text{for } v_l < v(k) < v_r,
\end{cases}
\]

(2.16)

where

\[
v_l = \frac{u(k-1) - m_t}{m_t} + c_t, \quad v_r = \frac{u(k-1) - m_r}{m_r} + c_r.
\]

Similarly, from (2.14), (2.15) with true backlash parameters, we obtain the exact discrete-time backlash inverse model \( v(k) = BI(u_d(k)) \) as

\[
v(k) = \begin{cases} 
  v(k-1) & \text{for } u_d(k) = u_d(k-1) \\
  \frac{u_d(k)}{m_t} + c_t & \text{for } u_d(k) < u_d(k-1) \\
  \frac{u_d(k)}{m_r} + c_r & \text{for } u_d(k) > u_d(k-1).
\end{cases}
\]

(2.17)

The discrete-time version of Lemma 2.1 states that the characteristic \( BI(\cdot) \) defined by (2.18) is the right inverse of the characteristic \( B(\cdot) \) defined by (2.16) such that

\[
u_d(k_0) = B(BI(u_d(k_0))) \Rightarrow B(BI(u_d(k_0))) = u_d(k_0),
\]

(2.19)

3. PARAMETRIZATION

To develop an adaptive law for updating the estimates \( \hat{m}_t(k), \hat{c}_r(k), \hat{m}_r(k), \hat{c}_r(k) \) of the backlash inverse parameters, it is crucial to express the backlash inverse error \( u(t) - u_d(t) \) in terms of a parametrizable part and a unparametrizable but bounded part.

To give a compact description for the adaptive backlash inverse, we introduce two indicator functions:

\[
\hat{\chi}_l(t) = \begin{cases} 
  0 & \text{for } u_d(t), v(t) \text{ on the upward side of } BI(\cdot) \\
  1 & \text{otherwise},
\end{cases}
\]

\[
\hat{\chi}_r(t) = \begin{cases} 
  0 & \text{for } u_d(t), v(t) \text{ on the downward side of } BI(\cdot) \\
  1 & \text{otherwise}.
\end{cases}
\]

From (3.1), (3.2), we have that

\[
\hat{\chi}_l(t) + \hat{\chi}_r(t) = 1, \quad \hat{\chi}_l(t)\hat{\chi}_r(t) = 0.
\]

Using (2.14), (2.15), (3.1)-(3.4), we express \( v(t) \) as:

\[
v(t) = (\hat{\chi}_l(t) + \hat{\chi}_r(t))v(t) = \frac{\hat{\chi}_l(t)}{m_l(t)}
\]

\[
\times (u_d(t) + \hat{m}_t(t)\hat{c}_r(t)) + \frac{\hat{\chi}_r(t)}{m_l(t)}
\]

\[
\times (u_d(t) + \hat{m}_r(t)\hat{c}_r(t)).
\]

(3.5)

Similarly, for the backlash \( B(\cdot) \), we introduce
three indicator functions \( \chi_i(t) \), \( \chi_i(t) \) and \( \chi_i(t) \), such that

\[
\chi_i(t) = \begin{cases} 
1 & \text{for } v(t), u(t) \text{ on the upward side of } B(\cdot) \\
0 & \text{otherwise},
\end{cases}
\]

(3.6)

\[
\chi_i(t) = \begin{cases} 
1 & \text{for } v(t), u(t) \text{ on the downward side of } B(\cdot) \\
0 & \text{otherwise},
\end{cases}
\]

(3.7)

\[
\chi_i(t) = \begin{cases} 
1 & \text{for } v(t), u(t) \text{ on an inner segment of } B(\cdot) \\
0 & \text{otherwise}.
\end{cases}
\]

(3.8)

Using the following obvious relationships:

\[
\chi_i(t) + \chi_i(t) + \chi_i(t) = 1, \tag{3.9}
\]

\[
\chi^2_i(t) = \chi_i(t), \quad \chi^2_i(t) = \chi_i(t), \quad \chi^2_i(t) = \chi_i(t),
\]

(3.10)

\[
\chi_i(t)\chi_i(t) = 0, \quad \chi_i(t)\chi_i(t) = 0, \quad \chi_i(t)\chi_i(t) = 0,
\]

(3.11)

we arrive at the compact expression for the output \( u(t) \) of \( B(\cdot) \):

\[
u(t) = (\chi_i(t) + \chi_i(t) + \chi_i(t))u(t)
= \chi_i(t)m_i(v(t) - c_i) + \chi_i(t)m_i(v(t) - c_i)
+ \chi_i(t)u_i,
\]

(3.12)

where \( u_i \) is a generic constant corresponding to the value of \( u(t) \) at any active inner segment characterized by

\[
\frac{u_i}{m_i} + c_i < v(t) < \frac{u_i}{m_i} + c_i. \tag{3.13}
\]

Multiplying both sides of (3.5) by \( \hat{\lambda}_i(t) \), using (4.4), we obtain

\[
\hat{\lambda}_i(t)u_a(t) = \hat{\lambda}_i(t)(\hat{m}_i(t)v(t) - \hat{m}_i(t)c_i(t)), \tag{3.14}
\]

and similarly we have

\[
\hat{m}_i(t)u_a(t) = \hat{\lambda}_i(t)(\hat{m}_i(t)v(t) - \hat{m}_i(t)c_i(t)). \tag{3.15}
\]

Using (3.3), (3.14), (3.15) we get the following expression of \( u_a(t) \):

\[
u_a(t) = (\hat{\lambda}_i(t) + \hat{\lambda}_i(t))u_a(t)
= \hat{\lambda}_i(t)(\hat{m}_i(t)v(t) - \hat{m}_i(t)c_i(t))
+ \hat{\lambda}_i(t)(\hat{m}_i(t)v(t) - \hat{m}_i(t)c_i(t)). \tag{3.16}
\]

From (3.12), (3.16), we have the following relationship between \( u(t) \) and \( u_a(t) \):

\[
u(t) = u_i(t) + \hat{\lambda}_i(t)(\hat{m}_i(t)v(t) - c_i)
- \hat{m}_i(t)v(t) + \hat{m}_i(t)c_i(t) + \hat{\lambda}_i(t)
\times (\hat{m}_i(t)v(t) - c_i) - \hat{m}_i(t)v(t) + \hat{m}_i(t)c_i(t))
+ d_0(t), \tag{3.17}
\]

where

\[
d_0(t) = (\chi_i(t) - \chi_i(t))(m_i(v(t) - c_i))
+ (\chi_i(t) - \chi_i(t))(m_i(v(t) - c_i))
+ \chi_i(t)u_i, \tag{3.18}
\]

which represents the unparametrizable part of the error between \( u(t) \) and \( u_a(t) \).

From (3.18), we see that \( d_0(t) \) is reduced to zero if \( \chi_i(t) - \chi_i(t) = \chi_i(t) - \chi_i(t) = \chi_i(t) = 0 \).

This condition is satisfied if \( \hat{m}_i = m_i, \quad \hat{m}_i = m_i, \quad \hat{c}_i = c_i, \) and \( \hat{c}_i = c_i, \) because, after initialization, the motion of \( v(t), u(t) \) will not be on any of the inner segments, and \( u(t), v(t) \) are on the upward (downward) side of \( B(\cdot) \) if and only if \( u_a(t) \) and \( v(t) \) are on the upward (downward) side of \( B(\cdot) \).

When the parameter estimation errors are present, the above condition is not satisfied so that \( d_0(t) \neq 0 \) in general. However, as we show next, \( d_0(t) \) is bounded.

**Proposition 3.1.** The unparametrizable part \( d_0(t) \) of the control error \( u(t) - u_a(t) \) is bounded for any \( t \geq 0 \).

**Proof.** There are three different cases to be examined:

1. if \( \chi_i(t) = 1, \chi_i(t) = \chi_i(t) = 0 \), then

\[
d_0(t) = \begin{cases} 
0 & \text{for } \hat{\lambda}_i(t) = 1, \hat{\lambda}_i(t) = 0 \\
(m_i - m_i)u(t) - m_i c_i + c_m, & \text{for } \hat{\lambda}_i(t) = 0, \hat{\lambda}_i(t) = 1;
\end{cases}
\]

(3.19)

2. if \( \chi_i(t) = 1, \chi_i(t) = \chi_i(t) = 0 \), then

\[
d_0(t) = \begin{cases} 
0 & \text{for } \hat{\lambda}_i(t) = 0, \hat{\lambda}_i(t) = 1 \\
(m_i - m_i)u(t) + m_i c_i - c_m, & \text{for } \hat{\lambda}_i(t) = 1, \hat{\lambda}_i(t) = 0;
\end{cases}
\]

(3.20)

3. if \( \chi_i(t) = 1, \chi_i(t) = \chi_i(t) = 0 \), then

\[
d_0(t) = \begin{cases} 
- m_i v(t) + c_m u_i, & \text{for } \hat{\lambda}_i(t) = 1, \hat{\lambda}_i(t) = 0 \\
- m_i v(t) + c_m u_i, & \text{for } \hat{\lambda}_i(t) = 0, \hat{\lambda}_i(t) = 1.
\end{cases}
\]

(3.21)

If \( m_i \neq m_i \), the intersection of the two lines of \( B(\cdot) \) must not be in the region of interest, by assumption. Therefore \( v(t) \) is bounded and \( u_i \) is given by

\[
u_i = m_i(v(t) - c_i), \quad c_i \in (c_i, c_i), \tag{3.22}
\]

or

\[
u_i = m_i(v(t) - c_i), \quad c_i \in (c_i, c_i).
\]
This shows that in all three cases, \( d_o(t) \) is bounded.

For the usual backlash characteristic, \( m_r = m_t = m \), and the expression (3.22) always holds. Therefore \( d_o(t) \) is bounded even if \( v(t) \) is not bounded. \( \nabla \)

Letting \( \hat{m}_r(t) = \hat{m}_t(t) = \hat{m}(t) \) and \( \hat{mc}(t) = \hat{m}(t)\hat{c}(t) \), we define

\[
\begin{align*}
\theta^* &= (mc_r, m, mc_t)^T, \\
\theta &= (mc_r, m, mc_t)^T, \\
\phi &= \theta(t) - \theta^*, \\
\omega &= (\hat{c}(t), -v(t), \hat{c}(t))^T.
\end{align*}
\]

(3.23)

Letting \( \hat{m}_r(t) = \hat{m}(t)\hat{c}(t) \), we define

\[
\begin{align*}
\theta^* &= (mc_r, m, mc_t)^T, \\
\theta &= (mc_r, m, mc_t)^T, \\
\phi &= \theta(t) - \theta^*, \\
\omega &= (\hat{c}(t), -v(t), \hat{c}(t))^T.
\end{align*}
\]

(3.24)

Finally, from (3.17), (3.23), (3.24), we obtain the parametrized expression for the backlash inverse error \( u(t) - u^*(t) \) in terms of the parameter error \( \phi_*(t) = \theta(t) - \theta^* \) with a bounded disturbance \( d_o(t) \):

\[
u(t) - u_d(t) = \phi(t)\omega(t) + d_o(t). \quad (3.25)
\]

This parametrization holds for the discrete-time case:

\[
u(k) - u_d(k) = \phi^*(k)\omega(k) + d_o(k). \quad (3.26)
\]

This expression will be important for our adaptive design.

4. AN INTRODUCTORY EXAMPLE

The purpose of this section is to give an introductory example of adaptive backlash inverse. To focus on the backlash problem, we consider that the linear part of the plant in the continuous-time form is \( G(D) = k_p/D \) where \( k_p \) is a known constant. For the backlash characteristic we assume that only its breakpoint parameter \( c_r = -c_t = c > 0 \) is unknown, while the slope \( m > 0 \) is known.

Our objective is to design an adaptive law to update the backlash inverse estimate and a control \( u_d(t) \) to stabilize the closed-loop system and make the plant output \( y(t) \) track a given reference signal \( y_m(t) \) which specifies the desired system behavior.

For a discrete-time control design, the linear part of the plant is given by

\[
y(t_{k+1}) = y(t_k) + k_p \int_{t_k}^{t_{k+1}} u(t) \, dt. \quad (4.1)
\]

When \( u(t) \) is piecewise constant and \( T^\Delta t_{k+1} - t_k > 0 \), then

\[
y(k + 1) = y(k) + Tk_pu(k). \quad (4.2)
\]

Define \( \hat{u}(k) = Tk_pu(k) = \hat{B}(u(k)), \) where \( \hat{B}(u(k)) = B(Tk_pm, Tk_pm, c, c; v(k)) \) is the modified backlash characteristic with slopes \( Tk_pm, Tk_pm \). With the non-unity gain \( Tk_p \) taken care of by the modified backlash model and \( \hat{u}(k) \) renamed as \( u(k) \), and from (4.2), the linear part of the plant becomes:

\[
y(k + 1) = y(k) + u(k). \quad (4.3)
\]

In the absence of backlash our design objective would be achieved by the controller

\[
u_d(k) = -y(k) + y_m(k + 1). \quad (4.4)
\]

In the presence of backlash we use this controller along with an adaptive scheme designed to update the backlash inverse on-line.

Since, by assumption, \( m \) is known and \( c_r = -c_t = c \), we let \( \hat{m}_r(t) = m, \hat{mc}(t) = -\hat{mc}(t) = mc(t) \). The backlash inverse error equation (3.26) becomes:

\[
u(k) - u_d(k) = \phi(k)\omega(k) + d_o(k). \quad (4.5)
\]

where \( \omega(k) = \hat{c}(k) - \hat{c}(k), \phi = \theta(k) - \theta^*, \theta(k) = \hat{mc}(k), \) and \( \theta = mc \).

For the tracking error \( e(k) = y(k) - y_m(k) \), we obtain from (4.3)-(4.5)

\[
e(k) = \theta(k - 1)\omega(k - 1) - \theta^*\omega(k - 1) + d_o(k - 1). \quad (4.6)
\]

and define the estimation error as:

\[
e(k) = e(k) + \theta(k)\omega(k - 1) - \theta(k - 1)\omega(k - 1).
\]

(4.7)

This is the implementable form of \( e(k) \) to be used in adaptive update laws. A simpler but unimplementable form of \( e(k) \) obtained from (4.6) and (4.7) is

\[
e(k) = \phi(k)\omega(k - 1) + d_o(k - 1). \quad (4.8)
\]

Using the implementable form of \( e(k) \), our update law for \( \theta(k) \) based on a gradient algorithm (Goodwin and Sin, 1984; Landau, 1990) with an initial estimate \( \theta(0) \) is

\[
\theta(k + 1) = \theta(k) - \frac{\gamma \omega(k - 1)e(k)}{1 + \omega^2(k - 1)} - \sigma(k)\theta(k), \quad 0 < \gamma < 1,
\]

where \( \sigma(k) \) is a “switching-\( \sigma \) signal” (Ioannou and Tsakalis, 1986). Its implementation requires a \textit{a priori} knowledge of an upper bound \( M \) on \( |\theta^*| \):

\[
\sigma(k) = \begin{cases} \sigma_0 & \text{for } |\theta(k)| > 2M, \\ 0 & \text{otherwise} \end{cases}, \quad 0 < \sigma_0 < \frac{1}{2}(1 - \gamma).
\]

(4.10)
The stability and tracking properties of the closed-loop system (4.3)-(4.5), (4.9) are:

**Proposition 4.1.** All signals in the closed-loop system are bounded and there exist $a_0 > 0$, $b_0 > 0$ such that

$$
\sum_{k=k_1}^{k_1+k_2} e^2(k) \leq a_0 \sum_{k=k_1+1}^{k_1+k_2+1} d^2(k) + b_0,
$$

(4.11)

for any $k_1 \geq 2$, $k_2 \geq 0$.

**Proof.** Using (4.8), (4.9) and introducing

$$
\tilde{e}(k) = \frac{e(k)}{1 + \omega^2(k-1)},
$$

we express the time increment of $V(k) = \phi^2(k)$ as

$$
V(k+1) - V(k) \leq -\sigma_0 \tilde{e}^2(k) - \sigma_0 \sigma(k) \tilde{e}^2(k) + \tilde{d}_0^2(k-1).
$$

(4.13)

This proves that $\phi(k)$ is bounded. By definition, $\omega(k)$ is bounded. Hence $e(k)$ in (4.6) is bounded, and so is $r(k)$. Finally, $u_r(k)$ in (4.4), $u(k)$ in (2.18), $u(k)$ in (2.16), and thus all closed-loop signals, are bounded.

Using (4.7), (4.9), we have that

$$
\tilde{e}^2(k) \leq 2\epsilon^2(k) + 2\omega^2(k-1)(\theta(k) - \theta(k-1))^2
$$

$$
\leq 2\epsilon^2(k) + 2\omega^2(k-1)\left(\frac{2\gamma^2\omega^2(k-2)}{1 + \omega^2(k-2)}\right)
$$

$$
\times \tilde{e}^2(k-1) + 2\sigma^2(k-1)\theta^2(k-1),
$$

(4.14)

which, in view of (4.12), (4.13) and the boundedness of $\omega(k)$, proves (4.11).

To evaluate the closed-loop system performance improvement achieved by the proposed adaptive backlash inverse, simulations were performed for the first-order plant (4.2). The plant parameters were taken as: $k_v = 3.7$ and $m = m = m = 1.3$. The parameter $c_r = -c = e = 1.25$ is unknown to the adaptive backlash inverse. The discrete-time time step $T = 0.1$ so the modified slope is $k_p T m = 0.4625.$

---

**Fig. 3.** Tracking errors for $y_m(t) = 10 \sin 1.26t$.

**Fig. 4.** Tracking errors for $y_m(t) = 10 \text{sgn} (\sin 2.2t)$ (square wave).
Three cases were studied for comparison: (1) only the controller (4.4) is applied, that is, no backlash inverse is implemented; (2) the controller (4.4) and a fixed backlash inverse (that is, the backlash inverse implemented with fixed parameter estimates) are applied; (3) the controller (4.4) and an adaptive backlash inverse are applied.

The system responses to $y_m(t) = 10 \sin 1.26t$ with $\tilde{c}(0) = 1.91$ are shown in Fig. 3, and the system responses to $y_m(t) = 10 \text{sgn} (\sin 2.2t)$ (square wave) with $\tilde{c}(0) = 0.59$ are shown in Fig. 4, where $\tilde{c}(t) = \frac{\tilde{m}(t)}{\tilde{m}(t)}$ and $\tilde{m}(t) = m$.

The simulation results show that the adaptive backlash inverse (case (3)) leads to major system performance improvements in all the cases of different initial conditions and different reference signals: in addition to the signal boundedness the adaptive scheme achieves convergence to zero of both tracking error and parameter error, while the control error also converges to zero because the parameter error does. The simulations results also show that a fixed backlash inverse whose parameter was either underestimated or overestimated (case (2)) is also useful; the tracking error is reduced while it is quite large in case (1) when no backlash inverse is used.

5. ADAPTIVE CONTROL DESIGN

We are now prepared to address the main problem of this paper: adaptive control design for an unknown discrete-time plant with unknown backlash at its input. Using $D$ to denote the z-transform variable or the advance operator, as the case may be, the unknown plant to be controlled is

$$y(k) = G(D)[u](k), \quad u(k) = B(v(k)),$$

$$G(D) = \frac{Z(D)}{R(D)}.$$  

Without loss of generality, the polynomials $Z(D)$ and $R(D)$ are assumed to be monic so that the high-frequency gain of $G(D)$ is one, and the actual high-frequency gain of the plant is represented by the slope $m$ of the backlash $B(.)$.

We make the following assumptions about the plant:

- (A1) $G(D)$ is minimum phase;
- (A2) the relative degree $n^*$ of $G(D)$ is known;
- (A3) the degree $n$ of $R(D)$ is known;
- (A4) $m \geq m_0$ for some known $m_0 > 0$, and $c_i \leq 0 \leq c_i$.

Our problem is to design an adaptive controller to achieve the tracking of a reference signal $y_m(k)$ by the plant output $y(k)$. To solve this problem we need a controller structure which not only achieves tracking when all the plant parameters are known but also results in a tracking error expression suitable for adaptive control design.

5.1. Controller structure

From the usual controller structures used in adaptive linear control we borrow the feedback part, that is, we pass the output $y(k)$ through a linear filter $\Theta^T_0(a^T(D), 1)^T$, where $a(D) = (D^{-n+1}, \ldots, D^{-1})^T$, and $\Theta_0 \in \mathbb{R}^n$. The new part of the controller is its forward part. It must incorporate the backlash inverse and, hence, cannot have a linear structure. In solving this new problem we still want to preserve the linear parametrization of the error equations, which will be the main tool of our adaptive design. A structure which meets this requirement is obtained by passing not only the control signal $v(k)$, but also the backlash inverse signals $\tilde{x}(k)$ and $\tilde{x}^*(k)$ through the filter formed of $a(D)$ and adjustable parameters $\theta_0, \theta_1 \in \mathbb{R}^{n-1}$.

Introducing the four regressors:

$$u_{or}(k) = a(O)[z](k), \quad u_{ot}(k) = a(O)[x](k),$$

$$u_{oo}(k) = a(D)[v](k), \quad u_{oy}(k) = a^T(D), 1)^T[y](k),$n

we propose the following linear-like structure of the nonlinear adaptive controller:

$$u_d(k) = \Theta^T_0 u_{or}(k) - \Theta^T_0 u_{ot}(k) + \Theta^T_0 u_{ov}(k) - y_m(k + n^*),$$

where $\tilde{c}(t)$ and $\tilde{x}(t)$ are obtained from the logic block $L$ which implements (3.1) and (3.2).

This controller is shown in Fig. 5 where $\tilde{c}(t)$ and $\tilde{x}(t)$ are obtained from the logic block $L$ which implements (3.1) and (3.2).

The output $u_d(k)$ of this controller is applied to the adaptive backlash inverse in order to generate the plant control input:

$$v(k) = B(\hat{u}_d(k)).$$

We now show that for this controller there exist matched values of the adjustable parameters resulting in exact tracking with internal stability.

![Fig. 5. The adaptive inverse controller structure.](image-url)
Lemma 5.1. There exist matched values \( \theta_{\ast}^{\ast}, \theta_{\ast}^{\ast}, \theta_{\ast}^{\ast}, \theta_{\ast}^{\ast}, \theta_{\ast}^{\ast}, \) of \( \theta_{\ast}, \theta_{\ast}, \theta_{\ast}, \theta_{\ast}, \theta_{\ast}, \) and \( \theta_{\ast}^{\ast} \) with which the controller (5.4) achieves the closed-loop global stability and tracking \( y(k + n*) = y_m(k + n^*) \).

Proof. We first express the matched values of the parameters \( \theta_{\ast}, \theta_{\ast}, \theta_{\ast}^{\ast} \) in the forward part of the controller in terms of the backlash parameters \( \theta_{\ast}^{\ast} = (m_{\ast}, m, m_{\ast}) \) multiplied by a parameter \( \theta_{\ast}^{\ast} \in \mathbb{R}^{n-1} \).

\[
\theta_{\ast}^{\ast} = -\theta_{\ast}^{\ast} m_{\ast}, \quad \theta_{\ast}^{\ast} = -\theta_{\ast}^{\ast} m, \quad \theta_{\ast}^{\ast} = -\theta_{\ast}^{\ast} m.
\]  

(5.6)

When then define \( \theta_{\ast}^{\ast} \) and the matched value \( \theta_{\ast}^{\ast} \) of the feedback parameter \( \theta_{\ast}^{\ast} \) as the solution of the Diophantine equation:

\[
\theta_{\ast}^{\ast} r_0(D) R(D) + \theta_{\ast}^{\ast} r_1(a^T(D), 1)^T Z(D) = R(D) - Z(D) D^{n-1}. \]  

(5.7)

Substituting the matched values in the controller (5.4) we see that its forward part is a linear parametrization of the nonlinear term \( \theta_{\ast}^{\ast} r_0(D)[ - \theta_{\ast}^{\ast} m_{\ast} ] (k) \), namely:

\[
\theta_{\ast}^{\ast} r_0(D)[ - \theta_{\ast}^{\ast} m_{\ast} ] (k) = \theta_{\ast}^{\ast} r_0(k) + \theta_{\ast}^{\ast} r_0(k). \]  

(5.8)

With the matched values and (5.8) the controller (5.4) has the form:

\[
u_d(k) = \theta_{\ast}^{\ast} r_0(D)[ - \theta_{\ast}^{\ast} m_{\ast} ] (k) + \theta_{\ast}^{\ast} r_0(k) \] 

(5.10)

On the other hand, when both sides of (5.7) are divided by \( R(D) \) and then operated on \( u(k) \), the resulting identity is

\[
u(k) = \theta_{\ast}^{\ast} r_0(D)[u(k) + \theta_{\ast}^{\ast} r_0(a^T(D), 1)^T y(k) + y_m(k + n^*) \] 

(5.11)

Equating \( u(k) \) of (5.11) with \( u_d(k) \) of (5.9) and using (5.10) prove \( y(k + n^*) = y_m(k + n^*) \). The closed-loop system is globally stable because with the matched values the closed-loop poles are zeros of \( D^{n-1} Z(D) \).

5.2. Adaptive scheme

Our major task now is to design an adaptive scheme to update the parameters of the backlash inverse (2.21) and the controller (5.4) to guarantee the signal boundedness for the closed-loop system. This task is achievable with the tools of adaptive linear control (Ioannou and Tsakalis, 1986; Egardt, 1979; Kreisselmeier and Anderson, 1986; Narendra and Annaswamy, 1989; Middleton et al., 1988; Praly, 1990).

One update law for \( \theta(k) \) with an initial estimate \( \theta(0) \) suggested by the form of the tracking error expression. For a more compact notation, we let

\[
\theta(k) = (\theta_{\ast}^{\ast}(k), \theta_{\ast}(k), \theta_{\ast}(k), \theta_{\ast}(k), \theta_{\ast}(k))^T, \\
\theta^* = (\theta_{\ast}^{\ast}, \theta_{\ast}^{\ast}, \theta_{\ast}^{\ast}, \theta_{\ast}^{\ast}, \theta_{\ast}^{\ast})^T. \]  

(5.12)

\[
\omega(k) = (\omega_{\ast}(k), \omega_{\ast}(k), \omega_{\ast}(k), \omega_{\ast}(k), \omega_{\ast}(k))^T. \]  

(5.13)

Lemma 5.2. The controller (5.4) with arbitrary \( \theta_{\ast}(k), \theta_{\ast}(k), \theta_{\ast}(k), \theta_{\ast}(k), \theta_{\ast}(k) \) results in the tracking error consisting of a linear part \( \phi_T(k) \omega(k) \) and a bounded part \( d_1(k) \), that is:

\[
e(k) \triangleq y(k) - y_m(k) = \phi^T(k - n^*) \omega(k - n^*) + d_1(k), \phi(k) = \theta(k) - \theta^*, \]

(5.14)

\[
d_1(k) = d_0(k - n^*) - \theta_{\ast}^{\ast} r_0(D)[d_0(k - n^*)]. \]  

(5.15)

Proof. Recall from (3.16), (3.23) and (3.24) that the expression of the backlash inverse estimate \( \bar{B}(\cdot) \) is \( u_d(k) = -\theta_{\ast}^{\ast} r_0(k) \omega(k) \) so that (3.26) gives

\[u(k) = -\theta_{\ast}^{\ast} r_0(k) + d_0(k). \]  

(5.16)

Substituting (5.16) in the right side of (5.11) and (3.26) in the left side of (5.11), and using (5.8), we obtain

\[u_d(k) + \phi_{\ast}(k) \omega_{\ast}(k) + d_1(k) = \theta_{\ast}^{\ast} r_0(k) + y_m(k + n^*) + e(k + n^*) + \theta_{\ast}^{\ast} r_0(k) + \theta_{\ast}^{\ast} r_0(k) + \theta_{\ast}^{\ast} r_0(k)
\]

(5.17)

Finally, we substitute (5.17) into (5.4) to get (5.14). \( \nabla \)

We have thus obtained a tracking error equation with a linear parametrization and an unknown, but bounded, disturbance.

\section*{5.2. Adaptive scheme}

Our major task now is to design an adaptive scheme to update the parameters of the backlash inverse (2.21) and the controller (5.4) to guarantee the signal boundedness for the closed-loop system. This task is achievable with the tools of adaptive linear control (Ioannou and Tsakalis, 1986; Egardt, 1979; Kreisselmeier and Anderson, 1986; Narendra and Annaswamy, 1989; Middleton et al., 1988; Praly, 1990).

One update law for \( \theta(k) \) with an initial estimate \( \theta(0) \) suggested by the form of the tracking error equation (5.14) is

\[
\theta(k + 1) = \theta(k) - \frac{\gamma \omega(k - n^*) e(k)}{1 + \omega^T(k - n^*) \omega(k - n^*)} - \sigma(k) \theta(k), \quad 0 < \gamma < 1, \]

(5.18)

where \( e(k) \) is the estimation error:

\[
e(k) = e(k) + (\theta(k) - \theta(k - n^*))^T \omega(k - n^*), \]

(5.19)

and \( \sigma(k) \) is a “switching-S” modification (Ioannou and Tsakalis, 1986) whose implementation requires \textit{a priori} knowledge of an upper
bound $M$ on the Euclidean norm $\|\theta^*\|_2$ of $\theta^*$:
\[
\sigma(k) =
\begin{cases} 
\sigma_0 & \text{for } \|\theta(k)\|_2 > 2M, \\
0 & \text{otherwise}
\end{cases}
\]
(5.20)

Although not shown in (5.18) we use projection to ensure that $\tilde{m}(k) \geq m_0$ and $\tilde{m}_2(k) \leq \tilde{m}_2(C(k))$ to implement the adaptive backlash inverse (2.21).

This adaptive control scheme has the following stability and tracking properties.

**Theorem 5.1.** All signals in the closed-loop system are bounded and there exist $\sigma_0 > 0$, $\beta_0 > 0$ such that
\[
\sum_{k=0}^{k_2} e^2(k) \leq \sum_{k=k_1}^{k_1+\kappa^*} \tilde{d}_i(k) + \beta_0, 
\]
(5.21)
for $n_0 = 2n^* + n - 1$ and any $k_1 \geq n_0$, $k_2 \geq 0$.

**Proof.** The first part of the proof, which shows the boundedness of the update law, is standard. Substituting (5.14) in (5.19) results in
\[
e(k) = \phi(k)w(k - n^*) + d_1(k). 
\]
(5.22)
Using (5.18), (5.22) and introducing
\[
\tilde{e}(k) = e(k), \\
\tilde{d}_i(k) = d_i(k) 
\]
(5.23)
we express the time increment of $V(k) = \phi^T(k)\phi(k)$ as
\[
V(k + 1) - V(k) \leq -\sigma_0 \sigma(\kappa) \theta^T(k)\theta(k) + d_1(k). 
\]
(5.24)
This proves that $\phi(k) = \theta(k) - \theta^*$ is bounded. In view of (5.22), (5.23) and the boundedness of $d_1(k)$, this implies that $\tilde{e}(k)$ is also bounded.

In the second part of the proof we use a novel technique to show the closed-loop signal boundedness. Let us introduce
\[
\tilde{\omega}(k) = (\omega_n^T(k), \omega_s^T(k))^T, \\
\omega_n(k) = a(D)[u](k).
\]
(5.25)
It can be shown that there exist bounded sequences $F_i(k) \in R^{4n \times (2n-1)}$, $g_i(k) \in R^{4n}$, $g_{3}(k) \in R^{2n-1}$ and constant $F_2 \in R^{(2n-1) \times 4n}$ such that
\[
\omega(k) = F_1(k)\tilde{\omega}(k) + g_1(k), \\
\omega(k) = F_2\omega(k) + g_3(k).
\]
(5.26)
Then we use (5.7) and the fact that
\[
y(k) = (Z(D)/R(D))[u](k) \\
to obtain
\]
\[
\theta^T_\omega \omega_n(k) + \theta^T_s \omega_s(k) = u(k) - y(k + n^*).
\]
(5.27)
Using this equality and the definition of $\tilde{\omega}(k)$, we express
\[
\tilde{\omega}(k + 1) = A^* \tilde{\omega}(k) + b^s y(k + n^*) = A^* \tilde{\omega}(k) \\
+ b^s Z(D) R(D)[u](k + n^*), 
\]
(5.28)
for some constant matrix $A^* \in R^{(2n-1) \times (2n-1)}$ and constant vector $b^* \in R^{2n-1}$. Since the first component of $\tilde{\omega}(k)$ is $u(k - n + 1)$, it follows that for $c^* = (1, 0, \ldots, 0)^T \in R^{2n-1}$,
\[
D^{-n+1}u(k) = c^T(DI - A^*)^{-1}b^* Z(D) R(D) D^{n^*}u(D), 
\]
(5.29)
which implies that
\[
c^* (DI - A^*)^{-1}b^* = \frac{R(D)}{Z(D) D^{n^*+1}}, \\
\det (DI - A^*) = D^{n+n^*+1}Z(D), 
\]
(5.30)
that is, $A^*$ is a stable matrix. Using (5.19) and the first equality of (5.28), we have
\[
\tilde{\omega}(k + 1) = A^* \tilde{\omega}(k) + b^s (y_m(k + n^*) + \epsilon(k + n^*) - \theta(k + n^*) - \theta(k))\omega(k). 
\]
(5.31)
From (5.30), all eigenvalues of $A^*$ are inside the unit circle of the complex plane. Therefore there exists a non-singular constant matrix $Q \in R^{(2n-1) \times (2n-1)}$ such that $||Q A^* Q^{-1}||_2 < 1$.
Define the vector norm $||\cdot||$ in $R^{2n-1}$ by: $||x|| = ||Qx||_2$. From (5.26), there exist constants $c_1 > 0$, $\tilde{c}_1 > 0$, $i = 1, 2$, such that for all $k \geq 0$
\[
||\tilde{\omega}(k)|| \leq c_1 ||\omega(k)||_2 + c_2, \\
||\omega(k)||_2 \leq \tilde{c}_1 ||\tilde{\omega}(k)|| + \tilde{c}_2. 
\]
(5.32)
(5.33)
Introducing $x(k) = |\tilde{\omega}(k + n^*)| + ||\theta(k + n^*) - \theta(k)||_2$, using (5.18), (5.24) and (5.32) it can be shown that ther exist constants $c_3 > 0$, $c_4 > 0$ such that
\[
\sum_{k=k_1}^{k_1+k_2} x^2(k) \leq c_3 \sum_{k=k_1}^{k_1+k_2} \frac{1}{(\tilde{\omega}(k)||^2+c_4). 
\]
(5.34)
It follows from (5.30), (5.31), (5.33), there exist constants $a_0 \in (0, 1)$, $c_5 > 0$, $c_6 > 0$ such that
\[
||\tilde{\omega}(k + 1)|| \leq (a_0 + c_5 x(k)) ||\tilde{\omega}(k)|| + c_6. 
\]
(5.35)
Substituting (5.19) in (5.31) and using (5.14), (5.26), we obtain
\[
\tilde{\omega}(k + 1) = (A^* + b^s \phi^T(k) F_1(k)) \tilde{\omega}(k) + g_3(k), 
\]
(5.36)
for some bounded sequence $g_3(k) \in R^{2n-1}$. Since $\phi(k), F_l(k) \text{ and } g_3(k)$ in (5.36) are bounded, $\bar{\omega}(k)$ grows at the most exponentially, that is, there exist constants $c_7 > 0, c_8 > 0$ such that for all $k \geq 0$

$$\|\bar{\omega}(k+1)\| \leq c_7 \|\bar{\omega}(k)\| + c_8.$$  \hspace{1cm} (5.37)

Now we show the boundedness of $\bar{\omega}(k)$ by contradiction. Assume that $\bar{\omega}(k)$ grows unboundedly. Then, in view of (5.37), given any $\delta_0 > 0$ and $k_2 > 0$, we can find $\delta \in (0, \delta_0)$ and $k_1 > 0$ such that

$$\|\bar{\omega}(k)\| \geq \frac{1}{\delta}, \quad k \in \{k_1 - n^*, \ldots, k_1 - 1\},$$  \hspace{1cm} (5.38)

$$\|\bar{\omega}(k)\| = \frac{1}{\delta}, \quad k = k_1,$$  \hspace{1cm} (5.39)

$$\|\bar{\omega}(k)\| \geq \frac{1}{\delta}, \quad k \in \{k_1 + 1, \ldots, k_1 + k_2 + 1\}.$$  \hspace{1cm} (5.40)

Therefore, for $j \in \{0, \ldots, k_2\}$, the state transition function of $\|\bar{\omega}(k+1)\| = (a_0 + c_2x(k))\|\bar{\omega}(k)\|$ satisfies

$$\sum_{k=k_1}^{k_1+j} (a_0 + c_2x(k)) \leq \left(a_0 + c_5 \sum_{k=k_1}^{k_1+j} x(k)\right)\|\bar{\omega}(k_1)\| + c_9,$$  \hspace{1cm} (5.41)

With any $\delta_0$ satisfying $a_0 + (c_2 \sqrt{c_3(n^* + 1)} \delta_0 < 1$, (5.35) and (5.41) imply that

$$\|\bar{\omega}(k_1 + j + 1)\| \leq \sum_{k=k_1}^{k_1+j} (a_0 + c_2x(k))\|\bar{\omega}(k_1)\| + c_9,$$  \hspace{1cm} (5.42)

for some constant $c_9 > 0$, and $\sum_{k=k_1}^{k_1+j} (a_0 + c_2x(k)) < \frac{1}{2}$ for any $j \geq j_1$ and some $j_1 \geq 0$. Hence, for

$$\delta_0 \in \left(0, \min \left\{ \frac{1 - a_0}{c_2 \sqrt{c_3(n^* + 1)}}, \frac{1}{2c_9} \right\} \right) \quad \text{and} \quad k_2 \geq j_1,$$

(5.42) implies that $\|\bar{\omega}(k_1 + j + 1)\| < 1/\delta$ for any $j \in \{j_1, \ldots, k_2\}$, which is a contradiction. Hence $\bar{\omega}(k)$ is bounded, and so is $\omega(k)$.

Next we use this fact to prove the bound given by (5.21). From (5.18) and (5.19), we first evaluate a bound on the sum of $\epsilon^2(k)$ in terms of $\epsilon^2(k)$ and $\sigma^2(k-1)\theta^2(k-1)\theta(k-1)$ and their past values. Since this sum contains $d_i(k)$ and its past values, we use (5.15) to express $d_i(k)$ in terms of $d_i(k - n^*)$ and its past values and finally obtain (5.21). V

We have thus shown that the adaptive law (5.18) ensures robustness with respect to the bounded disturbance $d_0(k)$. Another advantage of this update law is that the asymptotic tracking is achieved if the adaptive backlash inverse converges to the exact one, that is when $d_0(k)$ disappears for large $k$, see (5.21).

We have not yet shown that, in general, the tracking error $e(k)$ converges to zero. The dependence of $d_0(k)$ on the parameter error suggests that this will be so if the adaptive system has sufficiently rich signals. Extensive adaptive backlash inverse simulations showed that with rich signals the backlash inverse parameters converge to their matched values so that the asymptotic tracking is achieved.

6. CONCLUSIONS

This paper has presented what appear to be the first formulation and solution of the adaptive control problem for systems with backlash. To achieve this, we first demonstrated the right invertibility of a general backlash model and parametrized the error expression of a backlash inverse estimate needed for continuous and discrete-time implementation. We then introduced a new linear-like structure for a nonlinear controller capable of cancelling the effects of backlash. This controller structure makes it possible to obtain a linear error equation, with the effect of an inaccurate backlash inverse represented by a bounded disturbance. From this point on, a robust adaptive update law was designed to guarantee global signal boundedness. Simulation results showed major system performance improvements.

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