Problem 0 (For fun). M/G/1 Queue with Random-Sized Batch Arrivals. Consider the M/G/1 system with the difference that customers are arriving in batches according to a Poisson process with rate $\lambda$. Each batch has $n$ customers, where $n$ has a given distribution and is independent of customer service times. Show that the waiting time in queue is given by

$$W = \frac{\lambda \bar{n} X^2}{2(1 - \rho)} + \frac{X(n^2 - \bar{n})}{2\bar{n}(1 - \rho)}$$

(1)

where $X$ denotes service time.

Problem 1. In the M/G/1 system show that

- $P$(the system is empty) = $1 - \lambda \bar{X}$
- Average length of time between busy periods = $\frac{1}{\lambda}$
- Average length of busy period = $\frac{\bar{X}}{1 - \lambda \bar{X}}$
- Average number of customers served in a busy period = $\frac{1}{1 - \lambda \bar{X}}$

where $\bar{X}$ is average service time.

Problem 2. Consider a computer system with a CPU and one disk drive. After a burst at the CPU the job completes execution with probability 0.1 and request a disk I/O with probability 0.9. The time of a single CPU burst is exponentially distributed with mean 0.01 s. The disk service time is broken up into three phases: exponentially distributed seek time with mean 0.03 s, uniformly distributed latency time with mean 0.01 s, and a constant transfer time equal to 0.01 s. After a service completion at the disk, the job always requires a CPU burst. The average arrival rate of jobs is 0.8 job/s and the system does not have enough main memory to support multiprogramming. Solve for the average response time. In order to compute the mean and the variance of the service time distribution, you may need the following results on random sums.

[Random Sums]

We have considered sums of $N$ mutually independent random variable when $N$ is a fixed constant. Here we are interested in the case when $N$ itself is random variable that is independent of $X_k$. Given a list $X_1, X_2, ..., $ of mutually i.i.d. random variables with distribution function $F(x)$,
mean \( E[X] \) and variance \( Var[X] \), consider the random sum \( T = X_1 + X_2 + \ldots + X_N \). Here the pmf of the discrete random variable \( p_N(n) \) is assumed to be given. For a fixed value \( N = n \), the conditional expectation of \( T \) is easily obtained

\[
E[T|N=n] = \sum_i^n E[X_i] = nE[X]
\]  

(2)

Then, using the theorem of total expectation, we get

\[
E[T] = \sum_n nE[X]p_N(n) = E[X]\sum_n np_N(n) = E[X]E[N]
\]  

(3)

Similarly, we have

\[
Var[T] = E[T^2] - (E[T])^2 = E\{E[T^2|N=n]\} - (E[T])^2
\]  

\[
= E\{nVar[X] + n^2(E[X])^2\} - (E[T])^2 = Var[X]E[N] + (E[X])^2Var[N]
\]  

(4)

**Problem 3.** Persons arrive at a Xerox machine according to a Poisson process with rate one per minute. The number of copies to be made by each person is uniformly distributed between 1 and 10. Each copy requires 3 sec. Find the average waiting time in queue when:

(a) Each person uses the machine on a first-come first-serve basis.

(b) Persons with no more than 2 copies to make are given preemptive priority over other persons.

**Problem 4. Priority Systems with Multiple Servers.** Consider the priority systems assuming that there are \( m \) servers and that all priority classes have exponentially distributed service times with common mean \( 1/\mu \).

(a) Consider the nonpreemptive system. Show that the average queueing time is given by

\[
W_k = \frac{R}{(1-\rho_1-\ldots-\rho_{k-1})(1-\rho_1-\ldots-\rho_k)}
\]  

(5)

with \( R \), the mean residual time, is given by

\[
R = \frac{P_Q}{m\mu}
\]  

(6)

where \( P_Q \) is the steady-state probability of queueing given by the Erlang C formula.
[Here \( \rho_i = \lambda_i/(m\mu) \) and \( \rho = \sum_{i=1}^{n} \rho_i. \)]

(b) Consider the preemptive resume system. Argue that \( W_k() \), defined as the average time in queue averaged over the first \( k \) priority classes, is the same as for an M/M/m system with arrival rate \( \lambda_1 + \lambda_2 + \ldots + \lambda_k \) and mean service time \( 1/\mu \). Use Little’s Theorem to show that the average time in queue of a \( k^{th} \) priority class customer can be obtained recursively from

\[
W_1 = W_{(1)} \tag{7}
\]

\[
W_k = \frac{1}{\lambda_k} \left[ W_{(k)} \sum_{i=1}^{k} \lambda_i - W_{(k-1)} \sum_{i=1}^{k-1} \lambda_i \right], \quad k = 2, 3, \ldots, n \tag{8}
\]

[Erang C formula]

In a M/M/m system, the probability that an arrival will find all servers busy is

\[
P_Q = P\{Queueing\} = \frac{p_0(m\rho)^m}{m!(1-\rho)} \text{, where } p_0 \text{ is}
\]

\[
p_0 = \left[ \sum_{n=0}^{m-1} \frac{(m\rho)^n}{n!} + \frac{(m\rho)^m}{m!(1-\rho)} \right]^{-1} \tag{10}
\]