Homework 1a: (Due date: Feb. 12, 2004)

Question 1. Prove properties III and IV of a Poisson process (merging and splitting) listed in the class notes posted on the web site for the lecture on Stochastic processes. For the splitting case, just prove the random assignment case.

[Solution]

Merging:

The number of arrivals of the combined process in disjoint intervals is clearly independent, so we need to show that the number of arrivals in an interval is Poisson distributed, i.e.

\[ P\{A_1(t + \tau) + \ldots + A_k(t + \tau) - A_1(t) - \ldots - A_k(t) = n \} = e^{-(\lambda_1 + \ldots + \lambda_k)\tau} [\lambda_1 + \ldots + \lambda_k]^{n} / n! \]  

(1)

For simplicity let \( k = 2 \); a similar proof applies for \( k < 2 \). Then

\[ P\{A_1(t + \tau) + A_2(t + \tau) - A_1(t) - A_2(t) = n \} \]

(2)

\[
= \sum_{m=0}^{n} P\{A_1(t + \tau) - A_1(t) = m, A_2(t + \tau) - A_2(t) = n - m \} \\
= \sum_{m=0}^{n} P\{A_1(t + \tau) - A_1(t) = m \} P\{A_2(t + \tau) - A_2(t) = n - m \} \\
= \sum_{m=0}^{n} e^{-\lambda_1\tau} (\lambda_1\tau)^m / m! \cdot e^{-\lambda_2\tau} (\lambda_2\tau)^{n-m} / (n-m)! = e^{-\lambda_1\tau - \lambda_2\tau} [\lambda_1 + \lambda_2]^{n} / n! \]

Splitting:

Let us call the two transmission lines 1 and 2, and let \( N_1(t) \) and \( N_2(t) \) denote the respective numbers of packet arrivals in the interval \([0, t]\). Let also \( N(t) = N_1(t) + N_2(t) \). We calculate the joint probability

\[ P\{N_1(t) = n, N_2(t) = m\} = \sum_{k=0}^{\infty} P\{N_1(t) = n, N_2(t) = m | N(t) = k\} P\{N(t) = k\} \]

(3)

Since

\[ P\{N_1(t) = n, N_2(t) = m | N(t) = k\} = 0 \text{, when } k \neq n + m \]

(4)

we obtain
However, given that $n + m$ arrivals occurred, since each arrival has probability $p$ of being a line 1 arrival and probability $1 - p$ of them will be line 1 and $m$ of them will be line 2 arrivals, it follows that the probability $n$ of them will be line 1 and $m$ of them will be line 2 arrivals is the binomial probability

$$\binom{n + m}{n} p^n (1 - p)^m$$

Thus

$$P\{N_1(t) = n, N_2(t) = m\} = \binom{n + m}{n} p^n (1 - p)^m e^{-\lambda t(n + m)/(n + m)!} = e^{-\lambda t} \frac{(\lambda t)^n}{n!} e^{-\lambda t(1 - p)} \frac{(\lambda t(1 - p))^m}{m!}$$

Hence

$$P\{N_1(t) = n\} = \sum_{m=0}^{\infty} P\{N_1(t) = n, N_2(t) = m\}$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{m=0}^{\infty} \frac{(\lambda t(1 - p))^m}{m!} = e^{-\lambda t p} \frac{(\lambda t p)^n}{n!}$$

That is, $\{N_1((t), t \geq 0)\}$ is a Poisson process having rate $\lambda p$. Similarly we argue that $\{N_2((t), t \geq 0)\}$ is a Poisson process having rate $\lambda (1 - p)$. Finally from (7) it follows that the two process are independent since the joint distribution factors into the marginal distributions.

**Question 2.** Consider a packet stream whereby packets arrive according to a Poisson process with rate 10 packets/sec. If the interarrival time between any two packets is less than the transmission time of the first to arrive, the two packets are said to collide. Find the probabilities that a packet does not collide with either its predecessor or its successor, and that a packet does not collide with another packet assuming all packets have a transmission time of 20ms.

[Solution]
Fix a packet. Let \( r_1 \) and \( r_2 \) be the interarrival times between a packet and its immediate predecessor, and successor respectively as shown in the figure above. Let \( X_1 \) and \( X_2 \) be the lengths of the predecessor packet, and of the packet itself respectively. We have

\[
P\{\text{No collision w/predecessor or successor}\} = P\{r_1 > X_1, r_2 > X_2\} = P\{r_1 > X_1\} P\{r_2 > X_2\}
\]

(9)

\[
P\{\text{No collision with any other packet}\} = P_1 P\{r_2 > X_2\}
\]

(10)

where \( P_1 = P\{\text{No collision with all preceding packets}\} \).

For fixed packet lengths (=20ms)

\[
P\{r_1 > X_1\} = P\{r_2 > X_2\} = e^{-r_1 \times 20} = e^{-0.01 \times 20} = e^{-0.2}
\]

(11)

\[
P_1 = P\{r_1 \geq X_1\}
\]

(12)

Therefore the two probabilities of collision are both equal to \( e^{-0.4} = 0.67 \).